Extensions of $RT_0$ Topological Spaces of Fuzzy Sets

K.C. Chattopadhyay, H. Hazra and S.K. Samanta

Department of Mathematics, The University of Burdwan, Burdwan-713104, India
e-mail: kcchattopadhyay2009@gmail.com
Department of Mathematics, Bolpur College, Bolpur-731204, India
e-mail: h.hazra2010@gmail.com
Department of Mathematics, Visva Bharati, Santiniketan-731235, India
e-mail: syamal_123@yahoo.co.in

Abstract

The aim of this paper is to study extensions of $RT_0$ topological spaces of fuzzy sets. We also construct $RT_0$ principal extensions of $RT_0$ topological spaces of fuzzy sets with the $\alpha$-graded trace system for each $\alpha$ in $(0,1]$.

Keywords: Fuzzy topology, principal extension, remoted neighbourhood, $RT_0$ space, $\alpha$-graded trace system.

1 Introduction

In crisp topology, extension theory is a well developed theory (for references please see [2], [3], [9], [12], [19] and [20]). In fuzzy topology only some particular type of extensions such as compactifications, completions of fuzzy topological spaces and fuzzy uniform spaces have been studied in [15], [23], [24]. The fuzzyfication of general extension theory has been started by us in [5], where a concept of fuzzyfication of extensions of topological spaces of fuzzy sets is introduced and a method of construction of strongly $T_0$ principal extension of a strongly $T_0$ topological space of fuzzy sets is provided.

In this paper we study extension theory and provide a method of construction of $RT_0$ principal extension of an $RT_0$ topological space of fuzzy sets with the given $\alpha$-graded trace system for each $\alpha \in (0,1]$. In this setting for each $\alpha \in (0,1]$, we find an $RT_0$ principal extension of an $RT_0$ topological space...
(X, u) with the given α-graded trace system.
Chang [4] introduced the notion of fuzzy topological spaces. In this context it is worth noting that Chang’s fuzzy topology is in fact a crisp topology of fuzzy sets. In this paper Chang’s fuzzy topology will be referred to as topology of fuzzy sets. (X, u) will be called a topological space of fuzzy sets if X is a set and u is a Chang topology on it.
In Section 2, some known definitions and known results are given which will be used in the sequel.
In Section 3, a definition of $RT_0$ topological spaces of fuzzy sets is given. Some results concerning principal extensions have been established.
In Section 4, using the concepts and results of Section 3, we present a construction of $RT_0$ principal extension of $RT_0$ spaces with the given α-graded trace system.

2 Preliminaries

Let X be a nonempty set and Y be a nonempty subset of X. For a fuzzy set λ of Y, its natural extension $\lambda_{Y<X}$ is defined by $\lambda_{Y<X}(x) = \lambda(x)$ if $x \in Y$ and $\lambda_{Y<X}(x) = 0$ if $x \in X - Y$. When there is no chance of confusion, we shall use (for simplicity) the same symbol λ for $\lambda_{Y<X}$.

In what follows I will stand for $[0,1]$.

Definition 2.1 [14] Let (X, u) be a topological space of fuzzy sets. Then (X, u) is called $T_0$ if for any pair of distinct points $x, y \in X, \exists \lambda \in u$ such that $\lambda(x) \neq \lambda(y)$.

Definition 2.2 [4] Let (X, u) and (Y, v) be two topological spaces of fuzzy sets. A mapping $\eta : (X, u) \rightarrow (Y, v)$ is said to be continuous if $\eta^{-1}(\lambda) \in u, \forall \lambda \in v$.

Definition 2.3 [5] Let (X, u) and (Y, v) be two topological spaces of fuzzy sets. A mapping $\eta : (X, u) \rightarrow (Y, v)$ is said to be closed if $\eta(\mu) \in v', \forall \mu \in u'$, where $u'$ and $v'$ are the families of closed sets in (X, u) and (Y, v) respectively.

Definition 2.4 [11] Let (X, u) and (Y, v) be two topological spaces of fuzzy sets. A mapping $\eta : (X, u) \rightarrow (Y, v)$ is said to be open if $\eta(\lambda) \in v, \forall \lambda \in u$.

Definition 2.5 [11] A mapping $\eta : (X, u) \rightarrow (Y, v)$ is said to be a homeomorphism if $\eta$ is bijective, continuous and open (or closed).

Definition 2.6 [25] Let (X, u) be a topological space of fuzzy sets and λ be a fuzzy set in X. Then the closure of λ in (X, u) is defined by $cl_u \lambda = \land\{\mu \in u' : \mu \geq \lambda\}$.

When there is no chance of confusion regarding the role of u, $cl_u \lambda$ will simply be denoted by $cl \lambda$. 
Theorem 2.7 [16, 17] Let \((X, u)\) and \((Y, v)\) be two topological spaces of fuzzy sets and \(\eta : X \to Y\) be a mapping. Then \(\eta : (X, u) \to (Y, v)\) is continuous if and only if
\[
\eta(\text{cl}_u \lambda) \leq \text{cl}_v \eta(\lambda), \forall \lambda \in I^X.
\]

Theorem 2.8 [5] For a bijective mapping \(\eta : X \to Y, \eta : (X, u) \to (Y, v)\) is homeomorphism if and only if
\[
\eta(\text{cl}_u \lambda) = \text{cl}_v \eta(\lambda), \forall \lambda \in I^X.
\]

Definition 2.9 [25] Let \((X, u)\) be a topological space of fuzzy sets and \(A \subset X\). Let \(\lambda\) be a fuzzy set in \(X\). Then \(\lambda_A\) is a fuzzy set in \(A\) defined by
\[
\lambda_A(x) = \lambda(x), \forall x \in A.
\]
Define \(u_A = \{\lambda_A : \lambda \in u\}\). Then it is easily verified that \(u_A\) is a topology of fuzzy sets on \(A\) and \((A, u_A)\) is called a subspace of \((X, u)\).

Definition 2.10 [1] A fuzzy stack \(S\) on \(X\) is a subset of \(I^X\) such that \(\lambda \geq \mu \in S\) implies \(\lambda \in S\).

Definition 2.11 [1] A fuzzy grill \(G\) on \(X\) is a fuzzy stack on \(X\) such that
(i) \(\emptyset \notin G\),
(ii) \(\lambda \lor \mu \in G \Rightarrow \lambda \in G\) or \(\mu \in G\).

Remark 2.12 In this article fuzzy stacks and fuzzy grills as defined in [definitions 2.10 and 2.11] will be referred to as stacks of fuzzy sets and grills of fuzzy sets respectively.
A grill of fuzzy sets \(G\) is called proper if \(G \neq \emptyset\).

Definition 2.13 [5] A grill \(G\) of fuzzy sets on a topological space \((X, u)\) is said to be a c-grill of fuzzy sets if \(\text{cl}\lambda \in G \Rightarrow \lambda \in G\), \(\forall \lambda \in I^X\).

Definition 2.14 \(\forall f \in I^X\), we define \(Z(f)\) to be the subset \(\{x \in X : f(x) = 0\}\) of \(X\) which is called the zero-set of \(f\) in \(X\).

Remark 2.15 Here it is important to note that the symbol \(Z(f)\) has been used by Gillman and Jerison [13] for the zero-set of a real valued continuous function \(f\) on a topological space \(X\). In this article we use the same symbol for the zero-set of an arbitrary element \(f \in I^X\) for an arbitrary set \(X\).

Definition 2.16 [5] Let \((X, u)\) and \((Y, v)\) be two topological spaces of fuzzy sets and \(\eta : X \to Y\) be a mapping. Then \((\eta, (Y, v))\) is said to be an embedding of \((X, u)\) if \(\eta : (X, u) \to (\eta(X), v_\eta(X))\) is a homeomorphism.
Definition 2.17 [5] Let \((X, u)\) and \((Y, v)\) be two topological spaces of fuzzy sets and \(\eta : X \rightarrow Y\) be a mapping. Then \((\eta, (Y, v))\) is said to be an extension of \((X, u)\) if \((\eta, (Y, v))\) is an embedding and \(cl_v \eta(1_X) = 1_Y\) or equivalently \(cl_v \eta(X) = 1_Y\); subject to the assumption that \(1_{\eta(X)}\) is the fuzzy set in \(Y\) satisfying \(1_{\eta(X)}(y) = 1, \forall y \in \eta(X)\) and \(1_{\eta(X)}(y) = 0, \forall y \in Y - \eta(X)\).

Theorem 2.18 [5] If \(\eta : X \rightarrow Y\) is one-one and \((X, u)\), \((Y, v)\) are topological spaces of fuzzy sets, then \((\eta, (Y, v))\) is an extension of \((X, u)\) if and only if

\((i)\) \(\forall \lambda \in I_X, \eta(cl_u \lambda) = (cl_v \eta(\lambda)) \wedge \eta(1_X)\),

and

\((ii)\) \(cl_v \eta(1_X) = 1_Y\).

Definition 2.19 [5] Let \(E_1 = (\eta_1, (Y_1, v_1))\) and \(E_2 = (\eta_2, (Y_2, v_2))\) be two extensions of \((X, u)\). Then \(E_1\) is said to be greater than or equal to \(E_2\) (written as \(E_1 \geq E_2\)) if there is a continuous function \(f\) from \((Y_1, v_1)\) onto \((Y_2, v_2)\) such that \(f\eta_1 = \eta_2\).

Definition 2.20 [5] The extension \(E_1 = (\eta_1, (Y_1, v_1))\) is said to be equivalent to the extension \(E_2 = (\eta_2, (Y_2, v_2))\) (written as \(E_1 \approx E_2\)) if there is a homeomorphism \(h\) of \((Y_1, v_1)\) onto \((Y_2, v_2)\) such that \(h\eta_1 = \eta_2\).

Definition 2.21 [5] Let \((X, u)\) be a topological space of fuzzy sets and \(\mathcal{B}\) be a family of closed sets in \((X, u)\). Then \(\mathcal{B}\) is said to be a base for the closed sets in \((X, u)\) if each closed set in \((X, u)\) can be expressed as the infimum of a subfamily of \(\mathcal{B}\).

Theorem 2.22 [5] Let \(\mathcal{B} \subset I^X\) such that

\((i)\) \(0_X \in \mathcal{B}\),

\((ii)\) \(\forall \lambda_1, \lambda_2 \in \mathcal{B} \Rightarrow \lambda_1 \vee \lambda_2 \in \mathcal{B}\).

Then \(\mathcal{B}\) is a base for closed sets of some topology of fuzzy sets on \(X\).

Definition 2.23 [5] An extension \(E = (\eta, (Y, v))\) is said to be a principal extension of \((X, u)\) if \(\{cl_v \eta(\mu) : \mu \in I^X\}\) is a base for the closed sets in \((Y, v)\).

Definition 2.24 [18] A fuzzy point in a set \(X\) is a mapping \(\alpha_x : X \rightarrow I\), where \(x \in X, \alpha \in [0, 1]\) defined by \(\alpha_x(x) = \alpha\) and \(\alpha_x(y) = 0\) for \(y \neq x\). Here \(x\) is the support of the fuzzy point \(\alpha_x\) and \(\alpha\) its value.

A fuzzy point \(\alpha_x\) is said to belong to a fuzzy set \(\lambda\) in \(X\), denoted by \(\alpha_x \in \lambda\) if \(\alpha \leq \lambda(x)\).

Following [21] a definition of remoted neighbourhood of a fuzzy point is given below:
Definition 2.25 Let \((X, u)\) be a topological space of fuzzy sets and \(\alpha_x\) be a fuzzy point. Then \(\lambda \in u'\) is called a remoted neighbourhood of \(\alpha_x\) if \(\alpha_x \notin \lambda\). The set of all remoted neighbourhoods of \(\alpha_x\) is denoted by \(R_{\alpha_x}\).

Definition 2.26 [5] A topological space \((X, u)\) of fuzzy sets is called strongly \(T_0\) if for each pair of distinct points \(x, y \in X\), there is either a \(\lambda \in u\) such that \(\lambda(x) > 0\) and \(\lambda(y) = 0\) or there is a \(\mu \in u\) such that \(\mu(x) = 0\) and \(\mu(y) > 0\).

3 Some Basic Results on Extensions of Topological Spaces of Fuzzy Sets

We begin the section with the following definition.

Definition 3.1 A topological space \((X, u)\) of fuzzy sets is said to be \(RT_0\) if for each pair of distinct points \(x, y\) of \(X\) and for each \(\alpha \in (0, 1]\), \(\exists \lambda_x \in R_{\alpha_x}, \lambda_x \notin R_{\alpha_y}\) or \(\exists \mu_y \in R_{\alpha_y}, \mu_y \notin R_{\alpha_x}\).

Example 3.2 Let \(X = \{x, y\}\) and \(u = \{0_X, 1_X\} \cup \{x/\alpha, y/0\} : \alpha \in [0, 1)\). Then \(u' = \{1_X, 0_X\} \cup \{x/\alpha, y/0\} : \alpha \in (0, 1]\). Thus for each \(\alpha \in (0, 1]\), \(\exists \lambda_x = \{x/\alpha, y/0\} \in u'\) such that \(\alpha > 0 = \lambda_x(y)\) and \(\alpha \leq \lambda_x = \lambda_x(x)\). i.e., \(\alpha_y \notin \lambda_x\) and \(\alpha_x \notin \lambda_x\). i.e., \(\lambda_x \in R_{\alpha_y}\) and \(\lambda_x \notin R_{\alpha_x}\).

Therefore \((X, u)\) is an \(RT_0\)-topological space of fuzzy sets.

Theorem 3.3 If \((X, u)\) is \(RT_0\), then it is strongly \(T_0\).

Proof. Let \((X, u)\) be \(RT_0\) and \(x, y \in X\) such that \(x \neq y\). Then for each \(\alpha \in (0, 1]\), \(\exists \lambda_x \in R_{\alpha_x}, \lambda_x \notin R_{\alpha_y}\) or \(\exists \mu_y \in R_{\alpha_y}, \mu_y \notin R_{\alpha_x}\). Therefore for each \(\alpha \in (0, 1]\), \(\exists \lambda_x \in u'\) such that \(\alpha > \lambda_x(x), \alpha \leq \lambda_x(y)\) or \(\exists \mu_y \in u'\) such that \(\alpha > \mu_y(y), \alpha \leq \mu_y(x)\).

Thus for \(\alpha = 1\), \(\exists \lambda_1 \in u'\) such that \(\lambda_1(x) < 1\), \(\lambda_1(y) = 1\) or \(\exists \mu_1 \in u'\) such that \(\mu_1(y) < 1\), \(\mu_1(x) = 1\).

Taking \(\lambda'_1 = \gamma\) and \(\mu'_1 = \delta\) we have \(\exists \gamma \in u\) such that \(\gamma(x) > 0, \gamma(y) = 0\) or \(\exists \delta \in u\) such that \(\delta(y) > 0, \delta(x) = 0\).

Hence \((X, u)\) is strongly \(T_0\).

Note 3.4 But the converse of Theorem 3.3 is not true, which is justified by the following Example.

Example 3.5 Let \(X = \{x, y\}\), \(u = \{0_X, 1_X, \{x/0.4, y/0\}\}\).
Then \(u' = \{1_X, 0_X, \{x/0.6, y/1\}\}\).
If \(\alpha = 0.5\), then \(R_{\alpha_x} = R_{\alpha_y}\). Thus \((X, u)\) is not \(RT_0\).
But it is clear that \((X, u)\) is strongly \(T_0\).
Proof. Let \((X, u)\) be \(RT_0\). Then it is strongly \(T_0\) and hence it is \(T_0\). 

Note 3.7 But the converse of the theorem is not true, which is justified by the following example.

Example 3.8 Let \(X = \{x, y, z\}\), \(u = \{\tilde{0}_X, \tilde{1}_X, \{x/0.2, y/0.3, z/0.4\}\}\). Therefore \(u' = \{\tilde{1}_X, \tilde{0}_X, \{x/0.8, y/0.7, z/0.6\}\}\).

If \(\alpha = 0.5\), then \(R_{\alpha_x} = R_{\alpha_y} = R_{\alpha_z}\). Therefore \((X, u)\) is not \(RT_0\).

It is easy to check that \((X, u)\) is \(T_0\).

Definition 3.9 Let \((X, u)\) be a topological space of fuzzy sets. \(\forall x \in X, \forall \alpha \in (0, 1]\), define 
\[G_{\alpha_x} = \{\lambda \in I^X : x \overset{\alpha}{\in} cl \lambda\}\].

Theorem 3.10 Let \((X, u)\) be a topological space of fuzzy sets. Then \((X, u)\) is \(RT_0\) if and only if \(\forall x, y \in X, G_{\alpha_x} = G_{\alpha_y}\) for some \(\alpha \in (0, 1]\) imply \(x = y\).

Proof. Let \((X, u)\) be an \(RT_0\) topological space of fuzzy sets. Let \(\alpha \in (0, 1]\) and \(x, y \in X\) such that \(x \neq y\).

Since \((X, u)\) is \(RT_0\),
\[\exists \lambda \in R_{\alpha_x}, \lambda \notin R_{\alpha_y}, \quad (1)\]
or
\[\exists \mu \in R_{\alpha_y}, \mu \notin R_{\alpha_x}. \quad (2)\]

Without any loss of generality we assume that (1) holds.

Then \(\alpha_x \overset{\tilde{\lambda}}{\notin} cl \lambda, \alpha_y \overset{\tilde{\lambda}}{\notin} cl \lambda\), since \(\lambda\) is closed.

i.e., \(\lambda \in G_{\alpha_y}\) but \(\lambda \notin G_{\alpha_x}\).

Thus \(G_{\alpha_x} \neq G_{\alpha_y}\). Therefore the condition holds.

Conversely let the condition hold.

Let \(x, y \in X\) such that \(x \neq y\) and \(\alpha \in (0, 1]\).

Therefore \(G_{\alpha_x} \neq G_{\alpha_y}\).

Thus there exists \(\lambda_0 \in G_{\alpha_x}\) such that \(\lambda_0 \notin G_{\alpha_y}\) or there exists \(\mu_0 \in G_{\alpha_y}\) such that \(\mu_0 \notin G_{\alpha_x}\).

Therefore there exists \(\lambda_0 \in I^X\) such that \(\alpha_x \overset{\tilde{\lambda}}{\notin} cl \lambda_0, \alpha_y \overset{\tilde{\lambda}}{\notin} cl \lambda_0\) or there exists \(\mu_0 \in I^X\) such that \(\alpha_y \overset{\tilde{\lambda}}{\notin} cl \mu_0, \alpha_x \overset{\tilde{\lambda}}{\notin} cl \mu_0\).

Taking \(cl \lambda_0 = \gamma_0\) and \(cl \mu_0 = \delta_0\) we have \(\exists \gamma_0 \in R_{\alpha_y}, \gamma_0 \notin R_{\alpha_x}\) or \(\exists \delta_0 \in R_{\alpha_x}, \delta_0 \notin R_{\alpha_y}\).

Therefore \((X, u)\) is \(RT_0\). This completes the proof.

Theorem 3.11 Let \((X, u)\) be a topological space of fuzzy sets. Then \(\forall x \in X\) and \(\forall \alpha \in (0, 1]\), \(G_{\alpha_x}\) is a proper c- grill of fuzzy sets in \((X, u)\).
Proof. Let \( x \in X \) and \( \alpha \in (0, 1] \). Clearly \( \hat{0}_X \notin G_{\alpha_x} \).
Let \( \lambda, \mu \in I^X \). Then
\[
\lambda \geq \mu \in G_{\alpha_x} \Rightarrow \alpha_x \in \text{cl} \mu \leq \text{cl} \lambda \Rightarrow \lambda \in G_{\alpha_x}
\]
and
\[
\lambda \lor \mu \in G_{\alpha_x} \Rightarrow \alpha_x \in \text{cl} (\lambda \lor \mu) \Rightarrow \alpha_x \in \text{cl} (\text{cl} \lambda \lor \text{cl} \mu) \\
\Rightarrow \alpha_x \in \text{cl} \lambda \lor \alpha_x \in \text{cl} \mu \Rightarrow \lambda \in G_{\alpha_x}, \text{ or } \mu \in G_{\alpha_x}.
\]
Thus \( G_{\alpha_x} \) is a grill of fuzzy sets on \( X \).
Let \( \lambda \in I^X \). Then
\[
\text{cl} \lambda \in G_{\alpha_x} \Rightarrow \alpha_x \in \text{cl} (\text{cl} \lambda) \Rightarrow \alpha_x \in \text{cl} \lambda \Rightarrow \lambda \in G_{\alpha_x}.
\]
Therefore \( G_{\alpha_x} \) is a c-grill of fuzzy sets in \( (X, u) \).
Clearly \( \hat{1}_X \in G_{\alpha_x} \). Therefore \( G_{\alpha_x} \neq \phi \) and hence \( G_{\alpha_x} \) is proper.
Thus for each \( x \in X \) and for each \( \alpha \in (0, 1] \), \( G_{\alpha_x} \) is a proper c-grill of fuzzy sets in \( (X, u) \).

Definition 3.12 Let \( E = (\eta, (Y, v)) \) be an extension of \( (X, u) \). Let \( y \in Y \) and \( \alpha \in (0, 1] \). Define the trace \( T_{(\alpha_y, E)} \) of the point \( \alpha_y \) with respect to the extension \( E \) by
\[
T_{(\alpha_y, E)} = \{ \lambda \in I^X : \alpha_y \in \text{cl} \eta(\lambda) \}.
\]
When there is no chance of confusion, we shall simply write \( T_{\alpha_y} \) for \( T_{(\alpha_y, E)} \).
The \( \alpha \)-graded trace system \( X^E_{\alpha} \) of the extension \( E \) is defined by
\[
X^E_{\alpha} = \{ T_{\alpha_y} : y \in Y \}.
\]
Also define \( X^E_{[0,1]} \) by
\[
X^E_{[0,1]} = \{ T_{\alpha_y} : y \in Y, \alpha \in (0, 1] \}.
\]

Theorem 3.13 Let \( E = (\eta, (Y, v)) \) be an extension of \( (X, u) \). Then
(i) \( T_{\alpha_y} \) is a proper c-grill of fuzzy sets in \( (X, u), \forall y \in Y, \forall \alpha \in (0, 1] \).
(ii) \( T_{\eta(\alpha_x)} = G_{\alpha_x}, \forall x \in X, \forall \alpha \in (0, 1] \).

Proof. (i) Let \( y \in Y \) and \( \alpha \in (0, 1] \). Clearly \( \hat{0}_X \notin T_{\alpha_y} \).
Let \( \lambda, \mu \in I^X \) such that \( \lambda \geq \mu \in T_{\alpha_y} \). Then
\[
\alpha_y \in \text{cl} \eta(\mu) \leq \text{cl} \eta(\lambda).
\]
Therefore
\[
\forall \lambda, \mu \in I^X,
\lambda \lor \mu \in T_{\alpha_y} \Rightarrow \alpha_y \in \text{cl} \eta(\lambda \lor \mu) \Rightarrow \alpha_y \in \text{cl} \eta(\lambda) \lor \text{cl} \eta(\mu) \Rightarrow \alpha \leq \text{cl} \eta(\lambda \lor \mu) \Rightarrow \alpha \leq \text{cl} \eta(\lambda) \lor \text{cl} \eta(\mu) \Rightarrow \lambda \in T_{\alpha_y}, \text{ or } \mu \in T_{\alpha_y}.
\]
Also for \( \lambda \in I^X \),
\[
\text{cl} \lambda \in T_{\alpha_y} \Rightarrow \alpha \in \text{cl} \eta(\text{cl} \lambda) \Rightarrow \alpha \leq \text{cl} \eta(\text{cl} \lambda),
\]
\[
\text{since } \eta(\text{cl} \lambda) = (\text{cl} \eta(\lambda)) \vee (\hat{0}_X) \leq \text{cl} \eta(\lambda),
\]
\[
\Rightarrow \alpha \leq \text{cl} \eta(\lambda) \Rightarrow \alpha_y \in \text{cl} \eta(\lambda) \Rightarrow \lambda \in T_{\alpha_y}.
\]
Clearly $\tilde{1}_X \in T_{\alpha_\nu}$. Therefore $T_{\alpha_\nu} \neq \phi$.

Thus $T_{\alpha_\nu}$ is a proper c-grill of fuzzy sets in $(X,u)$, for each $y \in Y$ and for each $\alpha \in (0,1]$.

(ii) Let $x \in X$ and $\alpha \in (0,1]$. Let $\lambda \in I^X$. Then
\[
\begin{align*}
\lambda \in T_{\eta(\alpha_x)} & \iff \eta(\alpha_x) \in \text{cl}_v \eta(\lambda) \iff \alpha \in \text{cl}_v \eta(\lambda) \\
& \iff \alpha \leq (\text{cl}_v \eta(\lambda)) (\eta(x)) \iff \alpha \leq (\text{cl}_v \eta(\lambda) \land 1_{\eta(\lambda)} (\eta(x)) \\
& \iff \alpha \leq (\text{cl}_v \eta(\lambda) \land \eta(1_X)) (\eta(x)) \iff \alpha \leq \eta(\text{cl}_u \lambda)(\eta(x)) \\
& \iff \alpha \leq \text{cl}_u \lambda(x) \text{ (since } \eta \text{ is one-one }) \iff \lambda \in G_{\alpha_x}.
\end{align*}
\]

Thus $T_{\eta(\alpha_x)} = G_{\alpha_x}$.

**Theorem 3.14** If $E_1$ and $E_2$ be two equivalent extensions of $(X,u)$, then $X_{\alpha_1}^E = X_{\alpha_2}^E$ for each $\alpha \in (0,1]$ and hence $X_{(0,1]}^E = X_{(0,1]}^E$.

**Proof.** Let $E_1 = (\eta_1, (Y_1, v_1))$ and $E_2 = (\eta_2, (Y_2, v_2))$ be two equivalent extensions of $(X,u)$.

Then $\exists$ a homeomorphism $h$ of $(Y_1, v_1)$ onto $(Y_2, v_2)$ such that $ho\eta_1 = \eta_2$.

Let $y \in Y_1$, $\alpha \in (0,1]$ and $\lambda \in I^X$. Then
\[
\begin{align*}
\lambda \in T_{(\alpha_y, E_1)} & \iff \alpha_y \in \text{cl}_v \eta_1(\lambda) \iff h(\alpha_y) \in \text{cl}_v \eta_1(\lambda) \\
& \iff \alpha_{h(\lambda)} \in \text{cl}_v \eta_1(\lambda) \iff h(\eta_1(\lambda)) = ho\eta_1(\lambda) = \eta_2(\lambda). \\
& \iff \lambda \in T_{(\alpha_{h(\lambda)}, E_2)}.
\end{align*}
\]

Thus $T_{(\alpha_y, E_1)} = T_{(\alpha_{h(\lambda)}, E_2)}$.

Therefore $X_{\alpha_1}^E = \{T_{(\alpha_y, E_1)} : y \in Y_1\}$
\[
= \{T_{(\alpha_{h(\lambda)}, E_2)} : y \in Y_1\} = X_{\alpha_2}^E, \forall \alpha \in (0,1].
\]

Also
\[
X_{(0,1]}^E = \{T_{(\alpha_y, E_1)} : y \in Y_1, \alpha \in (0,1]\}
= \{T_{(\alpha_{h(\lambda)}, E_2)} : y \in Y_1, \alpha \in (0,1]\}
= \{T_{(\alpha_y, E_2)} : y \in Y_2, \alpha \in (0,1]\}
= X_{(0,1]}^E.
\]

**Note 3.15** Example is given below to show that the converse of Theorem 3.14 does not hold.

**Example 3.16** Let $X,Y,Z$ be three infinite sets such that $X \subset Y \subset Z$ and $|X| < |Y| < |Z|$, where $|X|$ denotes the cardinal number of the set $X$.

Let $u \subset I^Z$ be defined by
\[
\forall \lambda \in I^Z, \lambda \in u \text{ if and only if } \lambda = \tilde{0}_Z \text{ or } Z(\lambda) \text{ is finite}.
\]

Then it is clear that $u$ is a topology of fuzzy sets on $Z$.

Let $(X,u_X)$ and $(Y,u_Y)$ be subspaces of $(Z,u)$. Let $i : X \rightarrow Z$ be the inclusion map. Let $i$ also denote the inclusion map of $X$ into $Y$. 
Extensions of \(RT_0\) Topological

Obviously \(E_1 = (i, (Z, u))\) is an extension of \((X, u_X)\) and \(E_2 = (i, (Y, u_Y))\) is also an extension of \((X, u_X)\).

Note that for each \(x \in X\) and \(\forall \alpha \in (0, 1]\), \(T_{i(\alpha_x), E_1} = G_{\alpha_x} = T_{i(\alpha_x), E_2}\), i.e., \(T_{(\alpha, E_1)} = G_{\alpha_x} = T_{(\alpha, E_2)}\).

Let \(G^* = \{\lambda \in I^X : \lambda(a) = 1\text{ for infinitely many points } a \text{ of } X\}\).

Then it is easy to check that \(\forall \alpha \in (0, 1]\),
\[
T_{(\alpha, E_1)} = G^*, \quad \forall z \in Z - X \text{ and } T_{(\alpha, E_2)} = G^*, \quad \forall y \in Y - X.
\]

Hence \(X_{E_1}^\alpha = X_{E_2}^\alpha, \forall \alpha \in (0, 1]\) and hence \(X_{(0, 1]}^E = X_{(0, 1]}^E\).

But \(E_1 \not\approx E_2\), as \(|Y| < |Z|\).

**Theorem 3.17** For any extension \(E = (\eta, (Y, v))\) of \((X, u)\) and \(\forall y, z \in Y\),
\(G_{\alpha_y} \subset G_{\alpha_z}\) implies \(T_{\alpha_y} \subset T_{\alpha_z}\) for each \(\alpha \in (0, 1]\).

**Proof.** Let \(\alpha \in (0, 1]\) and \(y, z \in Y\) be such that \(G_{\alpha_y} \subset G_{\alpha_z}\).

Then \(\forall \mu \in I^X, \mu \in T_{\alpha_y} \Rightarrow \alpha_y \in \text{cl}_\alpha(y) \Rightarrow \eta(y) \in G_{\alpha_y} \Rightarrow \eta(y) \in G_{\alpha_z}\), since \(G_{\alpha_y} \subset G_{\alpha_z}\).

Thus \(T_{\alpha_y} \subset T_{\alpha_z}\).

**Note 3.18** An example is given below to show that the converse of the above theorem is not true.

**Example 3.19** Let \(Y\) be an infinite set. Let \(v \subset I^Y\) be defined by
\[\forall \lambda \in I^Y, \lambda \in v \quad \text{if and only if} \quad \lambda = \tilde{0}_Y \text{ or } Z(\lambda) \text{ is finite.}\]

Clearly \(v\) is a topology of fuzzy sets on \(Y\).

Let \(X\) be an infinite set such that \(X \subset Y\) and \(|Y - X| \geq 2\) and \(i : X \to Y\) be the inclusion map. Then it is easy to check that \((i, (Y, v))\) is an extension of \((X, v_X)\).

Let \(y, z (\neq y) \in Y - X\). Then it is clear that
\[
T_{\alpha_y} = \{\lambda \in I^X : \lambda(a) = 1 \text{ for infinitely many points } a \text{ of } X\}
\]
\[= T_{\alpha_z}, \forall \alpha \in (0, 1].\]

Choose \(\lambda, \mu \in I^Y\) such that
\[
\lambda(y) = 0.5, \lambda(z) = 0.6, \mu(y) = 0.6, \mu(z) = 0.3
\]
and both the sets \(\{a \in Y : \lambda(a) = 1\}\) and \(\{a \in Y : \mu(a) = 1\}\) are finite.

Then it is clear that \(\lambda \in G_{0.6_y}, \lambda \notin G_{0.6_y}\) and \(\mu \in G_{0.6_y}, \mu \notin G_{0.6_z}\).

Thus \(G_{0.6_y} \not\subset G_{0.6_y}\) and \(G_{0.6_y} \not\subset G_{0.6_z}\).

However the following result holds.

**Theorem 3.20** If \((\eta, (Y, v))\) is a principal extension of \((X, u)\), then \(\forall y, z \in Y\),
\(T_{\alpha_y} \subset T_{\alpha_z}\) if and only if \(G_{\alpha_y} \subset G_{\alpha_z}\) for each \(\alpha \in (0, 1]\).
Proof. ‘If part’ has already been proved above.

Let $\alpha \in (0, 1]$ and $y, z \in Y$ such that $T_{\alpha y} \subset T_{\alpha z}$.

Let $\lambda \in I^Y$ such that $\lambda \in G_{\alpha y}$. Then $\alpha_y \in cl_\lambda \lambda$.

Since
\[
\{ cl_\lambda \eta(\mu) : \mu \in I^X \}
\]

is a base for the closed sets in $(Y, v)$, $\alpha_y \in \{ cl_\lambda \eta(\mu) : \mu \in I^X, cl_\lambda \eta(\mu) \geq \lambda \}$.

Thus
\[
\alpha_y \in cl_\lambda \eta(\mu), \forall \mu \in I^X \text{ with } cl_\lambda \eta(\mu) \geq \lambda,
\]

and hence
\[
\mu \in T_{\alpha y}, \forall \mu \in I^X \text{ with } cl_\lambda \eta(\mu) \geq \lambda.
\]

Since $T_{\alpha y} \subset T_{\alpha z}, \mu \in T_{\alpha z} \forall \mu \in I^X \text{ with } cl_\lambda \eta(\mu) \geq \lambda$, which implies that
\[
\alpha_z \in \{ cl_\lambda \eta(\mu) : \mu \in I^X, cl_\lambda \eta(\mu) \geq \lambda \}.
\]

i.e., $\alpha_z \in cl_\lambda \lambda \text{ i.e., } \lambda \in G_{\alpha z}$.

Hence $G_{\alpha y} \subset G_{\alpha z}$.

The following corollary is an easy consequence of the above theorem.

**Corollary 3.21** If $(\eta, (Y, v))$ is a principal extension of $(X, u)$, then $\forall y, z \in Y, T_{\alpha y} = T_{\alpha z}$ if and only if $G_{\alpha y} = G_{\alpha z}$ for each $\alpha \in (0, 1]$.

**Theorem 3.22** If $(\eta, (Y, v))$ is a principal extension of $(X, u)$, then $(Y, v)$ is $RT_0$ if and only if
\[
\forall y, z \in Y, T_{\alpha y} = T_{\alpha z} \text{ for some } \alpha \in (0, 1] \implies y = z.
\]

Proof. Let $(Y, v)$ be $RT_0$. Let $y, z \in Y$ such that $T_{\alpha y} = T_{\alpha z}$ for some $\alpha \in (0, 1]$.

Thus $G_{\alpha y} = G_{\alpha z}$ and hence $y = z$ (see Theorem 3.10).

Conversely suppose that the condition holds.

i.e., $\forall y, z \in Y, T_{\alpha y} = T_{\alpha z}$ for some $\alpha \in (0, 1]$ implies $y = z$.

Let $G_{\alpha y} = G_{\alpha z}$ for some $\alpha \in (0, 1]$. Therefore by the above corollary we have
\[
T_{\alpha y} = T_{\alpha z}
\]

and hence by the given condition we have $y = z$.

Hence $(Y, v)$ is $RT_0$ (see Theorem 3.10).

4 Construction of $RT_0$ Principal Extension of an $RT_0$ Topological Space with the Given $\alpha$-Graded Trace System

In this section $(X, u)$ will be an $RT_0$ topological space of fuzzy sets and for each $\alpha \in (0, 1]$, $X^*_{\alpha}$ be a collection of proper c-grills of fuzzy sets in $(X, u)$ such that $G_{\alpha z} \in X^*_{\alpha}, \forall x \in X$.

Let $\alpha \in (0, 1]$. Define,
\[
f_{\alpha} : X \rightarrow X^*_{\alpha} \text{ by } f_{\alpha}(x) = G_{\alpha x}, \forall x \in X.
\]

In view of Theorem 3.10, it follows that $f_{\alpha}$ is one-one.

$\forall \lambda \in I^X$, define $\lambda_{\alpha}^* : X^*_{\alpha} \rightarrow I$ by the following:
and for $G \in X^*_\alpha - \{G_{\alpha x} : x \in X\}$,

$$
\lambda^c_\alpha(G_{\alpha x}) = \text{cl}_u \lambda(x), \forall x \in X
$$

Thus (i) \forall (\lambda \vee \mu)^c_\alpha(G_{\alpha x}) = \text{cl}_u(\lambda \vee \mu)(x) = (\text{cl}_u \lambda \vee \text{cl}_u \mu)(x) = \lambda^c_\alpha(G_{\alpha x}) \vee \mu^c_\alpha(G_{\alpha x}) = (\lambda^c_\alpha \vee \mu^c_\alpha)(G_{\alpha x}).

Also for $G \in X^*_\alpha - \{G_{\alpha x} : x \in X\}$,

$$
(\lambda \vee \mu)^c_\alpha(G) = (\lambda^c_\alpha \vee \mu^c_\alpha)(G),
$$

since $\lambda \vee \mu \in G$ if and only if $\lambda \in G$ or $\mu \in G$.

Thus $(\lambda \vee \mu)^c_\alpha = \lambda^c_\alpha \vee \mu^c_\alpha, \forall \lambda, \mu \in I^X$. Also $(\tilde{0}_X)^c_\alpha = \tilde{0}_{X^*}$. Thus \{\lambda^c_\alpha : \lambda \in I^X\} is a base for the closed sets of a topology $w_\alpha$ (say) of fuzzy sets on $X^*_\alpha$.

**Theorem 4.1** Let $\alpha \in (0,1]$ and $(X,u), (X^*_\alpha, w_\alpha)$ and the other symbols used below be same as above. Then

(i) \forall \lambda, \mu \in I^X, \lambda \leq \mu \Rightarrow \lambda^c_\alpha \leq \mu^c_\alpha .

(ii) \forall \lambda \in I^X, (\text{cl}_u \lambda)^c_\alpha = \lambda^c_\alpha .

(iii) \forall \lambda, \mu \in I^X, f_\alpha(\lambda) \leq \mu^c_\alpha \Leftrightarrow \text{cl}_u \lambda \leq \text{cl}_u \mu .

(iv) \forall \lambda \in I^X, \text{cl}_u f_\alpha(\lambda) = \lambda^c_\alpha .

(v) \text{cl}_u f_\alpha(\tilde{1}_X) = \tilde{1}_{X^*} .

(vi) \forall \lambda \in I^X, \text{cl}_u f_\alpha(\lambda) \wedge f_\alpha(\tilde{1}_X) = f_\alpha(\text{cl}_u \lambda) .

**Proof.** Let $\alpha \in (0,1]$. 

(i) \forall \lambda, \mu \in I^X ,

$$
\lambda \leq \mu \Rightarrow \lambda^c_\alpha(G) \leq \mu^c_\alpha(G), \forall G \in X^*_\alpha \Rightarrow \lambda^c_\alpha \leq \mu^c_\alpha .
$$

(ii) Let $\lambda \in I^X$. Then

$$(\text{cl}_u \lambda)^c_\alpha(G_{\alpha x}) = \text{cl}_u(\text{cl}_u \lambda)(x) = \text{cl}_u \lambda(x) = \lambda^c_\alpha(G_{\alpha x}), \forall x \in X$$

and clearly

$$(\text{cl}_u \lambda)^c_\alpha(G) = \lambda^c_\alpha(G) \text{ if } G \in X^*_\alpha - \{G_{\alpha x} : x \in X\} ,$$

since $G$ is a c-grill of fuzzy sets in $X$.

Thus $(\text{cl}_u \lambda)^c_\alpha(G) = \lambda^c_\alpha(G), \forall G \in X^*_\alpha$. Hence $(\text{cl}_u \lambda)^c_\alpha = \lambda^c_\alpha, \forall \lambda \in I^X$.

(iii) For $\lambda, \mu \in I^X ,$

$$
f_\alpha(\lambda) \leq \mu^c_\alpha \Leftrightarrow f_\alpha(\lambda)(G) \leq \mu^c_\alpha(G), \forall G \in X^*_\alpha \Leftrightarrow f_\alpha(\lambda)(G_{\alpha x}) \leq \mu^c_\alpha(G_{\alpha x}), \forall x \in X \Leftrightarrow f_\alpha(\lambda)(f_\alpha(x)) \leq \text{cl}_u \mu(x), \forall x \in X \Leftrightarrow \lambda(x) \leq \text{cl}_u \mu(x), \forall x \in X, \text{ since } f_\alpha \text{ is one-one.} \Leftrightarrow \lambda \leq \text{cl}_u \mu \Leftrightarrow \text{cl}_u \lambda \leq \text{cl}_u \mu .
$$

(iv) \forall \lambda \in I^X ,

...
Also if

\[
G \subset K.C.Chattopadhyay et al. (v) \quad cl_{\lambda}(\alpha) \quad (\forall \alpha, \mu, \mu \in I^X)
\]

is a base for the closed sets in \((X_\alpha^*, w_\alpha)\).

\[
= \wedge \{ \mu^\alpha_{\alpha} : cl_\alpha \lambda \leq cl_\alpha \mu, \mu \in I^X \}
\]

\[
= \wedge \{ (cl_\alpha \mu)_\alpha^\alpha : cl_\alpha \lambda \leq cl_\alpha \mu, \mu \in I^X \}
\]

\[
= (cl_\alpha \lambda)_\alpha^\alpha
\]

\[
= \chi_\alpha^\alpha.
\]

(v) \(cl_{\alpha} f_\alpha(\bar{1}_X) = (\bar{1}_X)_\alpha = \bar{1}_X \), since \((\bar{1}_X)_\alpha^\alpha(G) = 1 = \bar{1}_X \)(\forall G \in X_\alpha^*).

(vi) Let \(\lambda \in I^X\). Then \(\forall x \in X\),

\[
\left( (cl_\alpha \lambda)_\alpha \right)(G_{\alpha_\lambda}) = \left( \chi_\alpha^\alpha \wedge f_\alpha(\bar{1}_X) \right)(\lambda_{\alpha_\lambda})
\]

\[
= cl_\alpha \lambda(x) \wedge \bar{1}_X(x) = cl_\alpha \lambda(x) = f_\alpha(cl_\alpha \lambda)(f_\alpha(x))(\text{ since } f_\alpha \text{ is one-one})
\]

\[
= f_\alpha(cl_\alpha \lambda)(G_{\alpha_\lambda}).
\]

Also if \(G \in X_\alpha^* \setminus \{G_{\alpha_\lambda} : x \in X\}, then)

\[
\left( (cl_\alpha \lambda)_\alpha \right)(G) = \left( \chi_\alpha^\alpha \wedge f_\alpha(\bar{1}_X) \right)(G) = \chi_\alpha^\alpha(G) \wedge 0
\]

\[
= 0 = f_\alpha(cl_\alpha \lambda)(G).
\]

Thus \((cl_\alpha \lambda)_\alpha \wedge f_\alpha(\bar{1}_X) = f_\alpha(cl_\alpha \lambda).

This completes the proof.

**Remark 4.2** Since for each \(\alpha \in (0, 1]\), \(f_\alpha : X \to X_\alpha^*\) is one-one and \(\forall \lambda \in I^X, (cl_\alpha \lambda)_\alpha \wedge f_\alpha(\bar{1}_X) = f_\alpha(cl_\alpha \lambda)\) and \(cl_{\alpha} f_\alpha(\bar{1}_X) = \bar{1}_X \), it follows that \((f_\alpha, (X_\alpha^*, w_\alpha))\) is an extension of \((X, u)\) for each \(\alpha \in (0, 1]\).

Since for each \(\alpha \in (0, 1]\), \(\{ \chi_\alpha^\alpha : \lambda \in I^X \}\) is a base for the closed sets of \((X_\alpha^*, w_\alpha)\) and \(cl_\alpha \lambda \) is \(\chi_\alpha^\alpha, \forall \lambda \in I^X\), it follows that \((f_\alpha, (X_\alpha^*, w_\alpha))\) is a principal extension of \((X, u)\) for each \(\alpha \in (0, 1]\).

Note that \(\forall G_{\alpha_\lambda} \in X_\alpha^*\),

\[
T_{\alpha_\lambda} = \{ \mu \in I^X : G_{\alpha_\lambda} \subset cl_{\alpha} f_\alpha(\mu) \}
\]

\[
= \{ \mu \in I^X : (cl_{\alpha} f_\alpha(\mu)) \geq \alpha \}
\]

\[
= \{ \mu \in I^X : \mu_\alpha^\alpha(G_{\alpha_\lambda}) \geq \alpha \}
\]

\[
= \{ \mu \in I^X : cl_\alpha \mu(x) \geq \alpha \}
\]

\[
= \{ \mu \in I^X : \alpha_\lambda \subset cl_\alpha \mu \}
\]

\[
= G_{\alpha_\lambda}.
\]

Also if \(G \in X_\alpha^* \setminus \{G_{\alpha_\lambda} : x \in X\}, then)

\[
T_{\alpha} = \{ \mu \in I^X : \mu_\alpha^\alpha(G) \geq \alpha \}
\]

\[
= \{ \mu \in I^X : \mu_\alpha^\alpha(G) = 1 \}
\]

\[
= \{ \mu \in I^X : \mu \in G \}
\]

\[
= G.
\]

Thus \(T_{\alpha} = G, \forall G \in X_\alpha^*\).

Therefore \(X_\alpha^*\) is the \(\alpha\)-graded trace system of the extension \((f_\alpha, (X_\alpha^*, w_\alpha))\).

Also for each \(\alpha \in (0, 1]\) we have,

\[
\forall G_1, G_2 \in X_\alpha^*, T_{\alpha G_1} = T_{\alpha G_2} \Rightarrow G_1 = G_2,
\]

and hence \((X_\alpha^*, w_\alpha)\) is \(RT_0\) for each \(\alpha \in (0, 1]\).

Thus \((f_\alpha, (X_\alpha^*, w_\alpha))\) is an \(RT_0\) principal extension of \((X, u)\) with the given \(\alpha\)-graded trace system for each \(\alpha \in (0, 1]\).
Extensions of $RT_0$ Topological

**Notation 4.3** The extension $(f_\alpha, (X_\alpha^*, w_\alpha))$ will be denoted by $E_\alpha(X_\alpha^*)$. Thus $X_\alpha^{E_\alpha(X_\alpha^*)} = X_\alpha^*$.

5 Future Work

In [6], we introduced $T_0$ principal extensions of a $T_0$-topological spaces of fuzzy sets. In [8], we defined fuzzy conjoint compactness and fuzzy linkage compactness and established conditions on the trace systems which would ensure the fuzzy conjoint compactness and fuzzy linkage compactness of the $T_0$ principal fuzzy extensions. In [8], we also introduced basic fuzzy proximities, Lodato fuzzy proximities and eventually proved a theorem which establishes that there is a bijection between a class of Lodato fuzzy proximities compatible with a given strongly $T_1$-topological space of fuzzy sets $(X, c)$ and the class of strongly $T_1$ principal Type-II fuzzy linkage compactifications of $(X, c)$. Our aim is to achieve the similar result mentioned above in the $RT_0$ spaces.

ACKNOWLEDGEMENTS.
The present work is partially supported by Special Assistance Programme (SAP) of UGC, New Delhi, India.

References


