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Global Stabilization of a Fish Population System Using Criteria Sample of Stability

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Abstract

In this article, we show global asymptotic stability of an equilibrium point of a non-trivial non-linear system of dimension 4 that modeled fishery model using a simple criterion of stability polynomial. By using the direct Lyapunov method, we develop a simple design technique, which allows to study the stability of the nonlinear system of fishing. And main objective is to obtain simple sufficient conditions used to obtain the stability of our system around the equilibrium point of reference. The proposed approach is that sufficient conditions demonstrate the overall stability of the system under study can be presented as feasibility tests. The results are tested by numerical examples.

Keywords: *Continuous structured model, Hermite-Biehler theorem, Hurwitz polynomial, Lyapunov approach, Stability.*

1 Introduction

The exploitation of fish populations provides some classic examples of disastrous exploitation of a renewable resource. In practice, the management of a fishery is a decision with multiple objectives. One of the desirable objectives in the management of fish resources is the conservation of the fish population (perenniality of the stock). The formulation of good harvesting policies which take into account these objective recur the stability analysis study of harvested fish population models around the positive equilibrium point.
Over the past decades, several authors have studied the dynamic behavior of the stage structured fish population model. It is usually discussed as having stable equilibrium and unstable one. Its analysis was realized by usual tools of automatic control. Its origin can be traced back to Clarck [3] who applied the optimal control design approach to fisheries management. The continuous age structured model was built and studied by Touzaeu [24]. She showed the local stability of the nontrivial equilibrium under general biological assumptions, using the method of linearization and some results of cooperative systems (see [23]). In [4, 5, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18, 19] the authors built a quadratic Lyapunov function which permits stabilization of the continuous age structured model around the nontrivial steady states, under the condition that each predator lays more eggs than it consumes. They were interested in constructing a state feedback law that allowed to stabilize the system. In [4, 5, 6] we are interested in the design of a fishing strategy by constructing a linear feedback control law that permits to stabilize the studied system using Riccati equation. In [8] the problem of global state regulation via output feedback was investigated to study a structured fishing model, in order to stabilize its states around a non-trivial equilibrium. The two stages of the juvenile and adults ages of fish population are considered. In order to apply the tools of automatic control to this model, the fishing effort is used as a control term, the age classes as a states and the quantity of captured fish per unit of effort as a measured output. A Lyapunov function is adapted to study the stability and stabilization of the studied system around the nontrivial steady states. In our last previous work, we consider the problem of stabilizing a continuous fish population system via a state feedback control. The model considered is structured in n age classes and includes a non-linear stock-recruitment relationship. In our case, the variation of the fishing effort is used as a control term, the age classes as states and the quantity of captured fish per unit of effort as a measured output. The back stepping approach is employed to stabilize the studied system around a reference equilibrium point. Explicit expression of a bounded fishing effort and a Lyapunov function are given [6, 7, 16].
In this paper we attempt to solve the stability problem of the fish popula-

tion model. We first construct the Lyapunov function based on the structure of the studied system, that contains the cannibalism term which is expressed as a Lotka-Volterra predation term between the adults and the juveniles and the competition term that can be interpreted as an intra-stage competition for food and space in a limited environment. The we formulate some biologically meaningful sufficient conditions for the stability of positive stationary solutions of the studied system. The global asymptotic stability of the non trivial steady state follows from the classical Lyapunov technique. Finally a numerical simulation is taken to verify some of the key results we obtained.

2 The model and preliminaries

The modeling of the exploitation of biological resources like fisheries and forestries has gained importance in recent years. In order to stabilize and conserve fish population in marine ecosystems, various dynamic models for commercial fishing was proposed and analyzed by considering the economic and biological factors: Global models that gives a general vision of the stock, which is represented with a single variable [3, 21] and structured models that distinguishes between several stages (classes of ages, of size...) of the stock, the evolution of each one is described separately [4, 5, 8, 13, 14, 16, 19, 24].

In this work, we focus on the study of a structured model containing a stage of juvenile [24]. The exploited fish population model considered is structured in $(n+1)$ age classes ($n \geq 2$), where every stage i is described by the evolution of it's biomass x_i for $0 \leq i \leq n$. Stage 0 is the pre-recruits stage, each stage in the stock ($i = 1, \ldots, n$) is characterized by its fecundity, mortality and predation rates. In addition, a fishing effort is included in the global mortality term.

For $n = 4$, the dynamic of the fish population can be represented in the following system of differentials equations (see [23, 24]):

$$
\left\{
\begin{array}{lcl}
\dot{x}_0(t) & = & -\alpha_0 x_0(t) + \sum_{i=1}^{4} f_i l_i x_i(t) - \sum_{i=0}^{4} p_i x_i(t) x_0(t) \\
\dot{x}_1(t) & = & \alpha x_0(t) - (\alpha_1 + q_1 E)\, x_1(t) \\
\vdots & = & \vdots \\
\dot{x}_4(t) & = & \alpha x_3(t) - (\alpha_4 + q_4 E)\, x_4(t)
\end{array}
\right.
\tag{1}
$$

Where:

$x_i(t)$ and E are respectively the abundance of class i, and the fishing effort (in unit effort$\times$times^{-1}).

p_0 and p_i represent respectively the juvenile competition parameter and predation of class i on class 0 (in time$^{-1}\times$number^{-1}).

f_i and l_i are respectively the average number of recruits (no dimension) and

reproduction efficiency of class i (in time$^{-1}\times$number^{-1}).
q_i the relative catchability coefficient of class i (in unit effort^{-1}).
The linear aging coefficient α is supposed to be constant and defined as:
$\alpha_i = \alpha + M_i$ (in time^{-1}).
M_i is the natural mortality class rate i (in time^{-1}).

Origin is an equilibrium point corresponding to a depleted population and thus does have a great interest. Under the assumptions of nonlinearity and survival following (see [5, 23, 24]):

One non linearity at least must be considered:

$$\sum_0^4 p_i \neq 0. \tag{2}$$

The spawning coefficient must be big enough so as to avoid extinction:

$$\sum_1^4 f_i l_i \pi_i > \alpha_0 \text{ where } \pi_i = \frac{\alpha^i}{\prod_1^i (\alpha_j + q_j E)} \tag{3}$$

The system (1) admits the nontrivial equilibrium point x^* as follows:

$$x_0^* = \frac{\sum_1^4 f_i l_i \pi_i - \alpha_0}{p_0 + \sum_1^4 p_i \pi_i} \text{ and } x_i^* = \pi_i X_0^* \text{ for } i = 1, \ldots, 4.$$

Remark 2.1. *For biological view, the assumption (2) means that the model have at least a nonlinear term. The second assumption (3) translate a survival condition and means that the spawning rate must be greater than the mortality of the stage 0 and its aging coefficient.*

3 Global stability analysis

In the qualitative analysis of ordinary differential equations describing a biological system one relevant question is to determine conditions on the parameters of the model that ensures the global stability of an equilibrium point. In the direct method of Lyapunov the stability is granted by the existence of a positive definite function in a neighborhood of the equilibrium point, whose total time derivative is negative semi-definite. Such Lyapunov functions are usually difficult to obtain and no general recipe to construct them exists. To this end and to solve the problem of global stability of the fish population model (1), we are interested in constructing an appropriate Lyapunov function. Our investigation is based on previous works that concern the stability study of Lotka-Volterra systems (epidemic systems, predator, ecological systems,...) (see [2, 4, 5, 7]). More specifically, a Lyapunov function is adapted to our polynomial system, such that, its total time derivative is negative semi-definite.

3.1 Main Results

The methods described in this section poses the problem of stability as a simple polynomial positivity test of a univariate polynomial. In the literature we find works that address the links between stability defined by Hurwitz and Hermite-Biehler theorem (see [1, 12, 22]).
In [11], the authors have proven a new method deduced from the stability criterion of Hemite-Biehler which reduces to a simple algebraic criterion (for a polynomial of order m we have to check the positivity of a single polynomial).

Consider the property of interlacing zeros when a polynomial:

$$P(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \ldots + \alpha_1 z + \alpha_0 \text{ where } \alpha_m > 0$$

is written as follows:

$$P(z) = P^e\left(z^2\right) + zP^o\left(z^2\right)$$

where $m = 2k$ is even, P^e and P^o are then developed as follows:

$$\begin{array}{rcl} P^e\left(z^2\right) & = & \alpha_{2k}(z^2)^k + \ldots + \alpha_2 z^2 + \alpha_0 \\ P^o\left(z^2\right) & = & \alpha_{2k-1}(z^2)^{k-1} + \ldots + \alpha_3 z^2 + \alpha_1 \end{array}$$

and where $m = 2k+1$ is odd, P^e retains the same form as in the case where m is even, but P^o is then given by:

$$P^o\left(z^2\right) = \alpha_{2k+1}(z^2)^{k+1} + \ldots + \alpha_3 z^2 + \alpha_1$$

In the following theorem we cite a result of further theoretical practical criterion Hermite.

Theorem 3.1. *(see [11]) The real polynomial $P(z)$ is stable if and only if two conditions are verified:*

Roots of $P^e(-z^2)$ are real,

For all $z \in IR$,

$$W\left(P^e\left(-z^2\right), zP^o\left(-z^2\right)\right) = \begin{vmatrix} P^e(-z^2) & zP^o(-z^2) \\ \\ \dfrac{dP^e(-z^2)}{dz} & \dfrac{d(zP^o(-z^2))}{dz} \end{vmatrix} > 0$$

3.2 Global stability for a dynamical system of a fish population

In the following proposition, we will demonstrate the global stability of our system (1) under the assumptions (2) and (3), using the theorem 3.1.

Proposition 3.2. *For a constant fishing effort and under the (2) and (3).* x^* *is globally asymptotically stable equilibrium point for the system (1) under the following sufficient conditions:*

$$\begin{cases} \bar{p}_i \geq 0 \text{ for } i = 1, \dots, 4, \\ \\ \bar{p}_1 \geq 3 \max\left(\dfrac{\bar{p}_2}{p_0 X_2^*}, \sqrt{\dfrac{\bar{p}_3 X_1^*}{\alpha X_2^* X_3^*}}\right) \end{cases} \tag{4}$$

where: $\bar{p}_i = p_i - \dfrac{f_i l_i}{X_0^*}$ *for* $i = 1, \dots, 4.$

Proof 3.3. *We consider the following Lyapunov function:*

$$V(x) = \sum_0^4 \left(x_i - x_i^* - x_i^* \ln\left(\frac{x_i}{x_i^*}\right)\right)$$

It is clear that the function V *is positive definite and satisfactory:* $V\left(x^*\right) = 0$ *and* $V\left(x\right) > 0$ *for all* $x \neq x^*$.
The derivative of V *along the solutions of system (1) is:*

$$\dot{V}(x) = \left(x - x^*\right)^\top A \left(x - x^*\right) + \left(x - x^*\right)^\top A_1 \left(x - x^*\right).$$

where

$$A = \begin{pmatrix} p_0 & -\bar{p}_1 & -\bar{p}_2 & -\bar{p}_3 & -\bar{p}_4 \\ \frac{\alpha}{x_1^*} & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{x_2^*} & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{x_3^*} & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha}{x_4^*} & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha}{x_1^*} & -\overline{\alpha}_1 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{x_2^*} & -\overline{\alpha}_2 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{x_3^*} & -\overline{\alpha}_3 & 0 \\ 0 & 0 & 0 & \frac{\alpha}{x_4^*} & -\overline{\alpha}_4 \end{pmatrix},$$

$\bar{p}_i = p_i - \dfrac{f_i l_i}{x_0^*}$ *and* $\overline{\alpha}_i = \alpha_i + q_i E$ *for* $i = 1, \ldots, 4$.
For the second term of $\dot{V}(x)$ *is negative definite. It remains to show that the matrix* $(-A)$ *is positive definite, its characteristic polynomial can be written as:*

$$P(z) = z^4 + p_0 z^3 + \frac{\alpha \bar{p}_1}{X_1^*} z^2 + \frac{\alpha \bar{p}_2}{X_1^* X_2^*} z + \frac{\alpha \bar{p}_3}{X_1^* X_2^* X_3^*} \tag{5}$$

In this case we have $m = 4$ *so we will have:*

$$\begin{cases} P^e\left(-z^2\right) & = & z^4 - \dfrac{\alpha \bar{p}_1}{X_1^*} z^2 + \dfrac{\alpha \bar{p}_3}{X_1^* X_2^* X_3^*} \\ zP^o\left(-z^2\right) & = & -p_0 z^3 + \dfrac{\alpha \bar{p}_2}{X_1^* X_2^*} z \end{cases}$$

Firstly, we show that the polynomial $P^e\left(-z^2\right)$ *admits only real roots.*
In the expression of $P^e\left(-z^2\right) = 0$, *let* $s = z^2$, *we have the following equation:*

$$s^2 - \frac{\alpha \bar{p}_1}{X_1^*} s + \frac{\alpha \bar{p}_3}{X_1^* X_2^* X_3^*} = 0$$

and that this equation has solutions s *real, it suffices that its discriminant is positive. This is guaranteed under the condition (4). More solutions* s *are positive, which proves that the solutions* z *are all real.*

secondly, it was:

$$W(P^e(-z^2), zP^o(-z^2))$$

$$= \begin{vmatrix} P^e(-z^2) & zP^o(-z^2) \\ \frac{dP^e(-z^2)}{dz} & \frac{d(zP^o(-z^2))}{dz} \end{vmatrix}$$

$$= \begin{vmatrix} z^4 - \frac{\alpha\bar{p}_1}{x_1^*}z^2 + \frac{\alpha\bar{p}_3}{x_1^*x_2^*x_3^*} & -p_0z^3 + \frac{\alpha\bar{p}_2}{x_1^*x_2^*}z \\ 4z^3 - 2\frac{\alpha\bar{p}_1}{x_1^*}z & -3p_0z^2 + \frac{\alpha\bar{p}_2}{x_1^*x_2^*} \end{vmatrix}$$

$$= p_0z^6 + \left(-\frac{3\alpha\bar{p}_2}{x_1^*x_2^*} + \frac{\alpha p_0\bar{p}_1}{x_1^*}\right)z^4 + \left(\frac{\alpha^2\bar{p}_1\bar{p}_2}{x_1^{*2}x_2^*} - \frac{3\alpha p_0\bar{p}_3}{x_1^*x_2^*x_3^*}\right)z^2 + \frac{\alpha^2\bar{p}_2\bar{p}_3}{x_1^{*2}x_2^{*2}x_3^*}$$

$$= p_0z^6 + \frac{\alpha}{X_1^*}\left(-\frac{3\bar{p}_2}{x_2^*} + p_0\bar{p}_1\right)z^4 + \frac{\alpha}{x_1^*x_2^*}\left(\frac{\alpha\bar{p}_1\bar{p}_2}{x_1^*} - \frac{3p_0\bar{p}_3}{x_3^*}\right)z^2 + \frac{\alpha^2\bar{p}_2\bar{p}_3}{x_1^{*2}x_2^{*2}x_3^*}$$

Gold under the conditions (4), be reduced to prove that the coefficients: $\left(-\frac{3\bar{p}_2}{x_2^*} + p_0\bar{p}_1\right)$ *and* $\left(\frac{\alpha\bar{p}_1\bar{p}_2}{x_1^*} - \frac{3p_0\bar{p}_3}{x_3^*}\right)$ *a positive.*

Then $W(P^e(-z^2), zP^o(-z^2)) > 0$. *So after the theorem below, the polynomial* $P(z)$ *is stable.*

3.3 Numerical example

For the purpose of illustration, we review results obtained from the stabilization of a fishery characterized by the parameter values,in an appropriate units for $n = 4$: $p_0 = 1$, $p_1 = 0.5$, $p_2 = 0.9$, $p_3 = 1.8$, $p_4 = 1.4$, $f_1 = f_2 = f_3 = f_4 = 0.5$, $l_1 = 5$, $l_2 = 10$, $l_3 = 20$, $l_4 = 15$, $\alpha_0 = 2.6$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1.8$ and $\alpha = 1.8$.
The initial state and the constant fishing effort, the corresponding stable equilibrium is $x^* = (5.71, 4.57, 3.66, 1.54, 0.52)$.

It's clair that the parameter values given here satisfy the assumptions (2) and (3). Secondly the sufficient condition (4) is verified in this case.
The time evolution of the states x_i are shown in Fig. 1. From the solution curves, we infer that the system is globally stable about the interior equilibrium point.

Figure 1: **The time evolution of the states x_0, x_1, x_2, x_3 and x_4.**

4 Conclusion

An approach for global asymptotic stability of the polynomial fish population systems was presented in this paper. This approach is based on the construction of Lyapunov function. Sufficient conditions for the existence of such Lyapunov functions ensuring the stability of the nonlinear studied systems are proved and derived after considerable developments. To this end the Lyapunov function based on the functions used to a wide class of biological systems whas adapted to our model. The results obtained extend previous study [4, 5, 7, 8, 13, 23] focusing on the solution to the same problem solved using other tools of control engineering. The proposed method permits to prove that the condition $X_0^* \leq \frac{f_i l_i}{p_i}$ for $i = 1, \ldots, n$ on which the authors in [13] are based to study the stability of the non trivial steady state, is not necessary. The advantage of the proposed approach is that the derived conditions proving the stability of the studied systems. The simulation results demonstrate the effectiveness of our results.

References

[1] B. D. Anderson, E. Jury and M.Mansour, On robust Hurwitz polynomials, *IEEE TAC,* 32(10), (1987), pp. 909-913.

[2] V. Capasso, *Mathematical Structures of Epidemic Systems, Lecture Notes in Biomathematics 97*, Springer-Verlag, Heidelberg, (1993).

[3] C. W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resourses*, 2nd ed, Wiley-Interscience, New York, (1990).

[4] E. El Mazoudi, N. Elalami and M. Mrabti, "On the stabilisation of a system describing the dynamics of a fishery", *Africain conference on computer science and Applied mathematic*, CARI06, Coutonou Bénin, 289-296, (2006).

[5] E. El Mazoudi, N. Elalami and M. Mrabti, "Stabilization of a harvested fish population", *Multi conférences Computational Engineering in system Applications*, CESA06, Beijing Chine, v 1, 245-248, (2006).

[6] K. Hattaf, A. A. Lashari, Y. louartassi and N. Youssfi, "A delayed SIR epidemic model with a general incidence rate", *E. J. Qualitative Theory od Diff. Equ.*, No. 3, (2013), pp. 1-9.

[7] E. El Mazoudi, N. Elalami, A. Ouahbi, "Afeedback control law to stabilise a harvested fish population system," *Int. J. Modelling Identification and Control,* vol. 12, No.4, (2010), 341-347.

[8] E. El Mazoudi, N. Elalami and M. Mrabti, "Output feedback control for an exploited structured model of a fishing problem," *Journal of Biological Systems*, 16, (2008), 107-117.

[9] M. Esghir, Birecursive Interpolation Alghorithm: A Formalism for solving Systems of Linear Equations, *Applied Mathematical Sciences*, Vol. 7, no. 24, (2013), 1157-1169.

[10] M. Esghir, Y. louartassi and N. Elalami, New Alghorithm for computing Transfer Function from State Equation, *Applied Mathematical Sciences*, Vol. 7, no. 24, (2013), 1171-1182.

[11] C. Fonte and C. Delattre, "Un critère simple de stabilité polynomiale", *CIFA 2010*, (2010), Nancy, France.

[12] D. Henrion, "On convexity of the frequency response of a stable polynomial," *IEEE TAC*, 53(4), (2008), pp. 1062-1066.

[13] A. Ouahbi, A. Iggidr and M. El Bagdouri, "stabilization of an exploited fish population," *Systems Analysis Modelling Simulation*, 43, (2003), 513-524.

[14] Y. Louartassi, El H. El Mazoudi and N. Elalami, "A new generalization of lemma Gronwall-Bellman", *Applied Mathematical Sciences*, Vol. 6, no. 13-16, (2012), pp. 621-628.

[15] Y. Louartassi, El H. El Mazoudi and N. Elalami, "Global Stability of the Continuous Age Structured Fish Population System", *Advances in Theoretical and Applied Mathematics*, Volume 7, Number 2, (2012), pp. 125-132.

[16] Y. Louartassi, El H. El Mazoudi and N. Elalami, "A Sufficient Condition of a Polynomial Fish Population System to be Stable", *International Journal of Modeling, Simulation, and Scientific Computing*, Vol. 3 No. 3, (2012), pp 1-9, DOI No: 10.1142/S1793962312500122.

[17] Y. Louartassi, El H. El Mazoudi and N. Elalami, "Static output Feedback control for an age structured fish population system using generalization of Gronwall-Bellman lemma", *MAMERN11: 4th International Conference on Approximation Methods and Numerical Modelling in Environment and Natural Resources*, (2011), Saidia Morocco, May 23-26.

[18] Y. Louartassi, El H. El Mazoudi and N. Elalami, "Stability Analysis and Static Output Feedback Design for a Model the Fishery Resource with Reserve Area", *Applied Mathematical Sciences*, Vol. 6, no. 66, (2012), pp. 3289-3299.

[19] Y. Louartassi, El H. El Mazoudi and N. Elalami, "Static Output Feedback Stabilization of Non Homogeneous Dynamic System From the Fishery", *Advances and Applications in Mathematical Sciences*, Vol. 10(5), (2011), pp. 519-532.

[20] W. E. Ricker, "Stock and recruitement", *J. Fish. Res. Board Can*, 11, (1954), 559-623.

[21] M. B. Schaefer, "Some aspects of the dynamics of populations important to the management of the commercial marine fisheries," *Bull., Inter-American Tropical Tuna Comm.*, 1, (1954), 25-26.

[22] M. R. Stojic and D. D. Siljak, "Generalization of Hurwitz Nyquist and Mikhailov stability criteria," *IEEE TAC*, 10, (1965), pp. 250-254.

[23] S. Touzeau, "Modèles de Contrôle en Gestion des Pêches", *Thèse, Université de Nice-Sophia Antipolis*, France, (1997).

[24] Touzeau, S. and J. L. Gouzé, "On the stock-recruitment relationships in rish population models", *Environmental Modeling and Assessment*, 3, (1998), 87-93.