

Ecological Ammensal model with reserve for one species and harvesting for both the species at variable rates

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Abstract

The paper presents the study of a mathematical model of Ecological Ammensalism with reserve for Ammensal species and harvesting at variable rates for both the species under limited resources. A pair of first order non-linear coupled differential equations characterizes this model. All the four equilibrium points of the model have been identified and the criteria for the stability are discussed in all cases. Solutions for the linearized perturbed equations are derived and the results are validated accordingly.

AMS Classification: 92 D 25, 92 D 40.

Keywords: *Equilibrium points, Normal Steady state, harvesting, stability.*

1 Introduction

Ammensalism between two species involves one imbedding or restricting the success of the other without being affected in any manner (positively or negatively) due to the interaction. Research in theoretical ecology was initiated by Lotka [9] and Meyer [10] followed by several mathematicians and ecologists. They contributed their might to the growth of this area of knowledge as reported in the treatises of Kapur [6, 7]. N.C. srinivas [12] studied competitive eco-systems of two and three species with limited and unlimited resources. Later, Lakshminarayan and Pattabhi Ramacharyulu [8] studied Prey-predator ecological models with a partial cover for the prey and alternate food for the predator. Acharyulu [1-5] and Pattabhi Ramacharyulu investigated some results on the stability of an enemy and Ammensal species pair with various resources.

Mathematical modeling essentially consists of translating real world problems, solving the mathematical problems and interpreting the solutions in the language of real world. A real world problem in all its generality can seldom be translated in to a mathematical problem and even if it can be so translated, it may not be possible to solve the resulting mathematical problem in a satisfactory manner. Hence, it is necessary to 'simplify', 'idealize', or 'approximate' the problem by another problem, taken as a replica to the original problem and at the same time that can be meaningfully translated and solved mathematically. In this idealization process, all the essential features of the problem will be retained, giving up those features, which are not inevitable or so relevant to the situation under investigation.

It is very interesting to note that one of the important roles of mathematicians working in the areas such as life, medical and social sciences is to evolve new mathematical techniques dealing with complex situations, which will arise, in nature and that too in our routine. Many mathematicians like Rahnama Mohamad Bagher & Jahanshai Payman[11] and Wong Yee Leng & Siti Mariyam Shamsuddin [13] examined and developed various real life mathematical models with the help of soft computing techniques. Situations in life sciences are often quite complex. As such, one should have some insight into a situation before attempting to formulate a new mathematical model. Once a model is formulated, its consequences can be noticed by using a suitable mathematical technique and the results are compared with the observations. The discrepancies between theoretical conclusions based on the model and real world observations suggest further improvements to the model.

If no satisfactory comparison is noted between the conclusions based on the model and observations in the real world situation, the assumptions in model-making are either modified or another form / structure for the mathematical model is fabricated. Different mathematical soft computing techniques have been

employed in analyzing the model equations. This process would continue until a satisfactory model close to the real situation is derived. The mathematical and biological models complement each other in the process of understanding the reality. The mathematical model, structured on a biological base, will enable the prediction/quantitative estimation of population strength at subsequent instants of time. In the absence of a biological model, mathematical soft computing techniques will become more and more abstract and in general, making the whole treatment difficult to analyze in drawing meaningful conclusions. The absence of mathematical soft computing techniques make difficult to identify the general relevance of a particular biological model.

Notation adopted:

N_1, N_2 : The populations of the Ammensal (S_1) and enemy (S_2) species respectively at time t

a_1 : Natural growth rate of Ammensal (S_1) Species

a_2 : Natural growth rate of Enemy (S_2) Species

a_{11} : Rate of decrease of the Ammensal (S_1) due to insufficient food.

a_{12} : Rate of increase of the Ammensal (S_1) due to inhibition by the enemy.

a_{22} : Rate of decrease of the enemy (S_2) due to insufficient food.

h_1, h_2 : Respective harvesting rates of Ammensal (S_1) and Enemy (S_2) species

m_1 : (h_1/a_1) is decrease of S_1 due to harvesting.

m_2 : (h_2/a_2) is decrease of S_2 due to harvesting.

K_i : a_i/a_{ii} are the carrying capacities of N_i , $i = 1, 2$

α : a_{12}/a_{11} is the coefficient of Ammensalism.

b : The constant characterized by the cover which is provided for the Ammensal species ($0 < b < 1$)

The state variables N_1 and N_2 as well as the model parameters $a_1, a_2, a_{11}, a_{12}, a_{22}, h_1, h_2, m_1, m_2, K_1, K_2, \alpha, b$ are assumed to be non-negative constants. More over it is assumed that harvest rate h_i is less than the natural birth rate a_i of the species S_i ; $i=1,2$. This is reasonable since $h_i > a_i$ leads to the fast extinction of species S_i .

2 Basic Equations

The equation for the growth rate of Ammensal species (N_1) with a cover and harvesting at variable rate under limited resources is given by,

$$\frac{dN_1}{dt} = N_1 a_{11} [(1-m_1)K_1 - N_1 - (1-b)\alpha N_2] \quad (1)$$

The equations for the growth rate of enemy species (N_2) harvesting at variable rate under limited resources is given by,

$$\frac{dN_2}{dt} = a_{22} N_2 [K_2(1-m_2) - N_2] \quad (2)$$

3 Equilibrium Points

The system under investigation has four equilibrium sates:

$$\text{I. } \bar{N}_1 = (1-m_1)K_1; \bar{N}_2 = 0$$

[Ammensal survives and Enemy only is washed out state] (3)

$$\text{II. } \bar{N}_1 = 0; \bar{N}_2 = (1-m_2)K_2$$

[Enemy survives and Ammensal only is washed out state] (4)

$$\text{III. } \bar{N}_1 = (1-m_1) K_1 - (1-b) (1-m_2)\alpha K_2; \bar{N}_2 = (1-m_2)K_2.$$

[Co-existent state or normal steady state] (5)

This state would exist only when $\frac{(1-m_1)K_1}{(1-b)(1-m_2)\alpha} > K_2 = \text{carrying capacity of } S_2$

$$\text{IV. } \bar{N}_1 = 0; \bar{N}_2 = 0 \text{ [Fully washed out state].} \quad (6)$$

4 Stability of Equilibrium sates

4.1 Stability of the Equilibrium State-I

The corresponding linearised perturbed equations are

$$\frac{dU_1}{dt} = -(1-m_1) a_1 U_1 - (1-m_1)(1-b)a_{12}K_1U_2$$

and $\frac{dU_2}{dt} = (1-m_2) a_2 U_2$ (7)

The characteristic equation for this system is

$$(\lambda + (1-m_1) a_1) (\lambda - (1-m_2) a_2) = 0 \quad .$$

The roots of which are $-(1-m_1)a_1$, $(1-m_2)a_2$ and hence the steady state is **unstable**.

By solving (6) we get

$$U_1 = U_{10} e^{-(1-m_1)a_1 t} + \frac{(1-m_1)(1-b)a_1 a_{12} U_{20}}{a_{11}((1-m_1)a_1 + (1-m_2)a_2)} [e^{-(1-m_1)a_1 t} - e^{(1-m_2)a_2 t}] \quad (8)$$

$$\text{and } U_2 = U_{20} e^{-(1-m_2)a_2 t} \quad (9)$$

The solution curves are illustrated in Fig.1 and Fig.2 and the observations are presented accordingly.

Case 1: when $U_{10} < U_{20}$

The enemy (S_2) outnumbers the Ammensal (S_1) in natural growth rate as well as in its initial population strength where as the Ammensal declines and the enemy is going away from the equilibrium point, as shown in Fig.1.

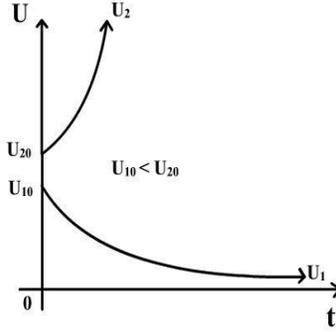


Fig .1

Case 2: when $U_{10} > U_{20}$

The Ammensal (S_1) dominates over the enemy (S_2) up to the time -instant after the the enemy dominates over the Ammensal, and the Ammensal is gradually decreasing as shown in Fig.2.

$$t^* = \frac{1}{(1-m_1)a_1 + (1-m_2)a_2} \log \frac{U_{10}((1-m_1)a_1 + (1-m_2)a_2)a_{11} + (1-m_1)(1-b)a_1 a_{12} U_{20}}{U_{20}((1-m_1)a_1 + (1-m_2)a_2)a_{11} + (1-m_2)(1-b)a_2 a_{12}}$$

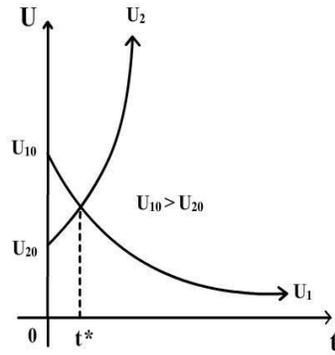


Fig .2

4.1.1 Trajectories of Perturbed Species

The trajectories (solution curves of (8) and (9)) in $U_1 - U_2$ plane are given by,

$$x + P_y = (1 + P) y \frac{(1-m_1) a_1}{(1-m_2) a_2} \tag{10}$$

where $x = \frac{U_1}{U_{10}}$, $y = \frac{U_2}{U_{20}}$ and $P = \frac{(1-m_1)(1-m_2)a_1 a_2 U_{20}}{((1-m_1)a_1 + (1-m_2)a_2)a_{11}U_{10}}$, as illustrated in Fig .3.

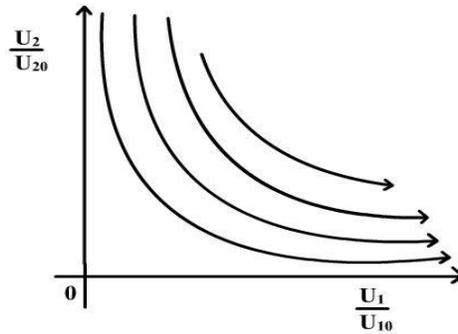


Fig .3

4.2 Stability of the Equilibrium State-II

The corresponding Linearised perturbed equations are

$$\frac{dU_1}{dt} = (1-m_1) a_1 U_1 - K_2(1-m_2)(1-b)a_{12} U_1, \tag{11}$$

$$\frac{dU_2}{dt} = -(1-m_2) a_2 U_2 \quad (12)$$

the characteristic equation is

$$\left(\lambda - \left((1-m_1)a_1 - (1-m_2)(1-b)K_2 a_{12} \right) \right) \left(\lambda + (1-m_2)a_2 \right) = 0 \quad (13)$$

The roots of this equation are $(1-m_1)a_1 - (1-m_2)(1-b)K_2 a_{12}$; $-(1-m_2) a_2$;

Case (1): if $\frac{(1-m_1)K_1}{(1-b)(1-m_2)\alpha} > K_2$

In this case, clearly one of the two roots is positive. Hence, the steady state is **unstable**.

We have obtained the solution as $U_1 = U_{10} e^{\left((1-m_1)a_1 - (1-m_2)(1-b)K_2 a_{12} \right) t}$; (14)

$U_2 = U_{20} e^{-a_2(1-m_2)t}$ (15)

The solution curves are illustrated in Fig.4 and Fig.5 and the conclusions are given accordingly.

Sub Case 1: when $U_{10} < U_{20}$

The enemy (S_2) outnumberes the Ammensal (S_1) up to the time instant after Ammensal outnumberes the enemy and the enemy is gradually declining (Fig .4).

$$t^* = \frac{a_{22}}{\left((1-m_1)a_1 + (1-m_2)a_2 \right) a_{22} - (1-b)(1-m_2)a_2 a_{12}} \log \left(\frac{U_{20}}{U_{10}} \right)$$

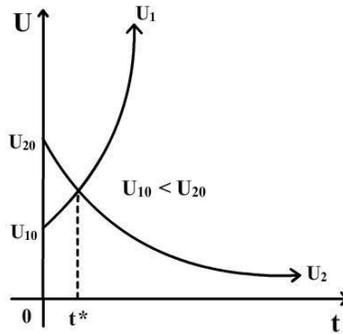


Fig.4

Sub Case 2 : $U_{10} > U_{20}$

The Ammensal(S_1) reigns over the enemy(S_2) in natural growth rate as well as in its initial population strength where as the enemy decays as shown in Fig .5.

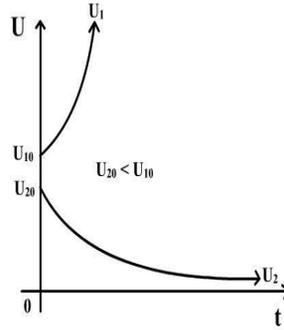


Fig .5

Case (2): If $\frac{K_1(1-m_1)}{\alpha(1-b)(1-m_2)} < K_2$

In this case, both the roots are negative; hence, the steady state is **stable**. We have obtained the solution as

$$U_1 = U_{10} e^{((1-m_1)a_1 - (1-m_2)(1-b)K_2a_{12})t}; U_2 = U_{10} e^{-(1-m_2)a_2t} \tag{16}$$

The solution curves are illustrated in **Fig (6)** and Fig (7) and the conclusions are presented.

Sub Case 1: $U_{10} > U_{20}$

The Ammensal (S_1) species prevails the enemy (S_2) species in the natural growth rate as well as in its initial population strength. It is also evident that both the species are converging asymptotically to the equilibrium point as shown in Fig 6.

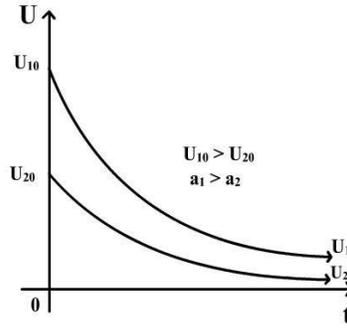


Fig .6

Sub Case 2: $U_{10} < U_{20}$

The Ammensal (S_1) species dominates over the enemy (S_2) species in the natural growth rate but its initial strength is less than that of enemy species. The enemy dominates over the Ammensal up to the time –instant after both Ammensal (N_1) and the enemy(N_2) gradually declining (Fig. 7).

$$t^* = \frac{a_{22}}{(a_1(1-m_1)+a_2(1-m_2))a_{22} - a_2(1-m_2)a_{12}} \log\left(\frac{U_{20}}{U_{10}}\right)$$

Here the time t^* may be called as the dominance time of Ammensal over the Enemy.

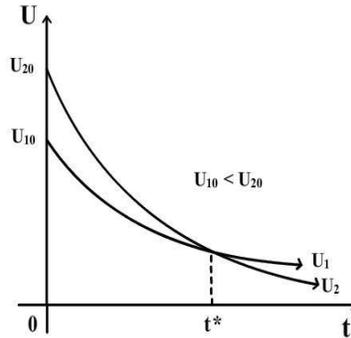


Fig .7

4. 2.1 Trajectories of Perturbed species

The trajectories (solutions curves of (8)) in $U_1 - U_2$ plane are given by

$$\left(\frac{U_1}{U_{10}}\right)^{\frac{a_{22}}{a_1(1-m_1)+a_2(1-m_2))a_{22} - a_2(1-m_2)a_{12}}} \cdot \left(\frac{U_2}{U_{20}}\right)^{1/a_2(1-m_2)} = 1; \quad (17)$$

Let $M = \frac{a_{22}}{(1-m_1)a_1 a_{22} - (1-m_2)(1-b)a_2 a_{12}}$ and are shown in Fig. 8 and Fig. 9.

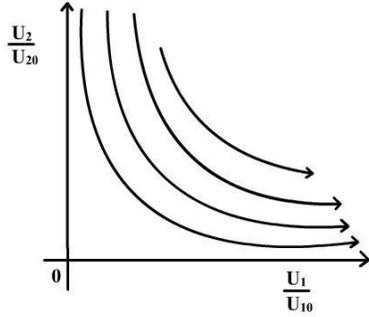


Fig.8
 (If $\frac{K_1(1-m_1)}{\alpha(1-b)(1-m_2)} > K_2$)

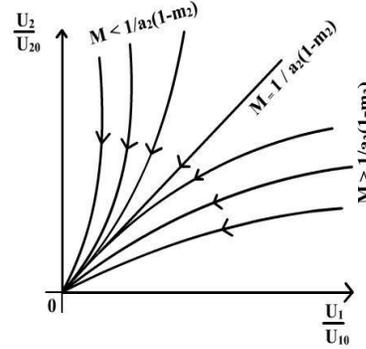


Fig.9
 (If $\frac{K_1(1-m_1)}{\alpha(1-b)(1-m_2)} < K_2$)

4.3 Stability of the Equilibrium State-III

The corresponding linearized perturbed equations are

$$\frac{dU_1}{dt} = - (a_1 - K_2(1-m_2)a_{12}) U_1 - a_{12} ((1-m_1)a_1 - (1-m_2)(1-\alpha)K_2 a_{12}) U_2 \tag{18}$$

$$\frac{dU_2}{dt} = - a_2 (1-m_2)U_2 \tag{19}$$

The characteristic equation is

$$(\lambda + ((1-m_1)a_1 - (1-m_2)(1-b)K_2 a_{12})) (\lambda + (1-m_2)a_2) = 0, \tag{20}$$

and the roots of this equation are $-(a_1 - K_2 a_{12})$ and $-a_2$, both are negative.

Hence the state is **stable**.

From (18) and (19), we get

$$U_1 = U_{10} e^{-((1-m_1)a_1 - (1-m_2)(1-b)a_2)a_{12}t} + \frac{U_{20}(1-b)a_{12}((1-m_1)a_1 a_{22} - (1-m_2)(1-b)a_2 a_{12})}{a_{11}[(1-m_1)a_1 - (1-m_2)a_2]a_{22} - (1-m_2)(1-b)a_2 a_{12}} \left[e^{-((1-m_1)a_1 - (1-m_2)(1-b)a_2)a_{12}t} - e^{-(1-m_2)a_2 t} \right] \quad (21)$$

and

$$U_2 = U_{20} e^{-(1-m_2)a_2 t} \quad (22)$$

The solution curves are illustrated in Fig.10 and Fig.11.

Case (i) when $U_{10} < U_{20}$

The enemy (S_2) species always rule over the Ammensal(S_1). It is evident that both the species are converging asymptotically to the equilibrium point as shown in Fig .10.

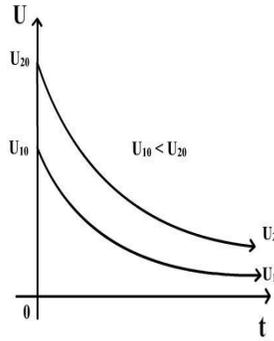


Fig .10

Case(i) when $U_{10} > U_{20}$

The Ammensal (S_1) species always dominates over the enemy(S_2) species. However, both converge asymptotically to the equilibrium point as shown in Fig.11.

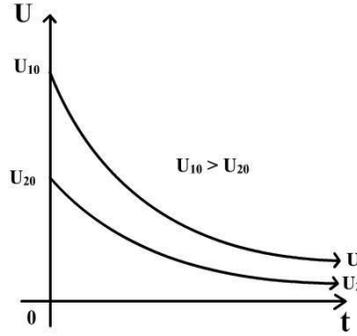


Fig .11

4.3.1 Trajectories of Perturbed Species

The trajectories (solution curves of (21) and (22)) in the $U_1 - U_2$ plane are given by

$$x = \left(\frac{MU_{20}}{U_{10}} + 1 \right) y^{(K_{11} - (1 - m_1) a_1)} - y \frac{MU_{20}}{a_{11} U_{10}} \quad (23)$$

where $x = \frac{U_1}{U_{10}}, y = \frac{U_2}{U_{20}},$

$$M = \left(\frac{((1 - m_1)(1 - b)a_1 a_{12} a_{22} - (1 - m_2)(1 - b)a_2 a_{12})}{a_{22}((1 - m_1)a_1 - (1 - m_2)a_2) - (1 - m_2)(1 - b)a_2 a_{12}} \right), \text{ and}$$

$$K_{11} = \left(\frac{(1 - m_2)(1 - b)a_2 a_{12}}{a_{22}} \right).$$

Fig.12 illustrates the above behaviour.

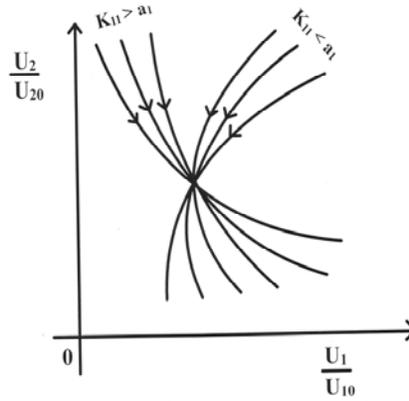


Fig .12

4.4 Stability of the Equilibrium state-IV

After linearization, we get

$$\frac{dU_1}{dt} = (1-m_1)a_1U_1 \quad \text{and} \quad \frac{dU_2}{dt} = (1-m_2)a_2U_2 \quad (24)$$

The characteristic Equation is $(\lambda - (1-m_1)a_1)(\lambda - (1-m_2)a_2) = 0$, whose roots a_1, a_2 are both positive. Hence, the steady state is **unstable**.

$$\text{By solving (24), we get } U_1 = U_{10} e^{a_1(1-m_1)t} ; U_2 = U_{20} e^{a_2(1-m_2)t} \quad (25)$$

where U_{10}, U_{20} are initial values of U_1, U_2 respectively. The solution curves are illustrated in the following four cases.

Case 1: $a_1 > a_2$ and $U_{10} > U_{20}$

The Ammensal (S_1) outnumbers the enemy (S_2) in natural growth rate as well as in its initial population strength. Both are diverging from the equilibrium point as shown in Fig .13.

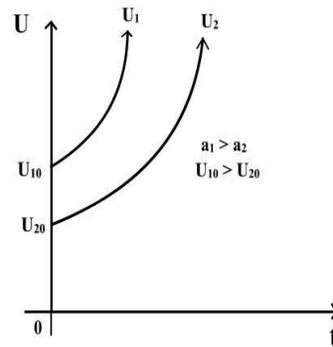


Fig .13

Case 2: $a_1 > a_2$ and $U_{10} < U_{20}$

The Ammensal(S₁) species outnumbers the enemy(S₂) species in the natural growth rate but its initial strength is less than that of the enemy species. The enemy outnumbers the Ammensal till the time- instant after the Ammensal outnumbers the enemy (Fig. 14).

$$t^* = \frac{1}{(1-m_2)a_2 - (1-m_1)a_1} \log \left(\frac{U_{10}}{U_{20}} \right).$$

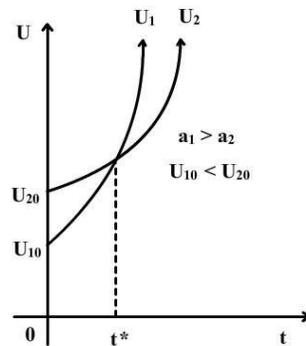


Fig .14

Case 3: $a_1 < a_2$ and $U_{10} > U_{20}$

The enemy(S₂) species prevails the Ammensal(S₁) species in the natural growth rate but its initial strength is less than that of Ammensal species .The Ammensal prevails the enemy till the time-instant (Fig. 15).

$$t^* = \frac{1}{(1-m_2)a_2 - (1-m_1)a_1} \log \left(\frac{U_{10}}{U_{20}} \right).$$

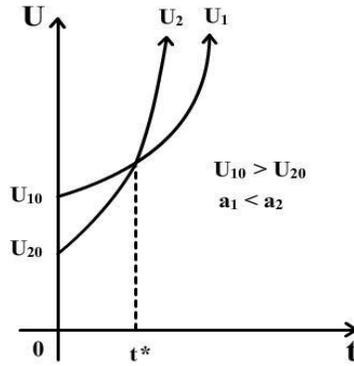


Fig .15

Case 4: $a_1 < a_2$ and $U_{10} < U_{20}$

The enemy (S_2 species) reigns over the Ammensal (S_1) species in the natural growth rate as well as in its initial population strength. The enemy continues to reign the Ammensal species and also we notice that both the species are diverging from the equilibrium point as shown in Fig.16.

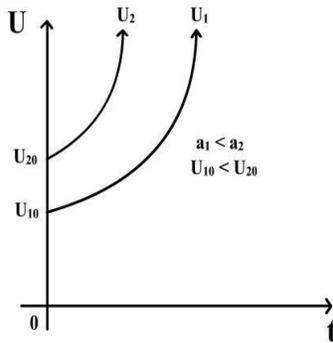


Fig.16

4.4.1 Trajectories of Perturbed Species

Further the trajectories (solution curves of (25)) in the $U_1 - U_2$ plane are given by

$$\left(\frac{U_1}{U_{10}}\right)^{a_2(1-m_2)} = \left(\frac{U_2}{U_{20}}\right)^{a_1(1-m_1)} \quad (26)$$

and these are illustrated in Fig. 17.

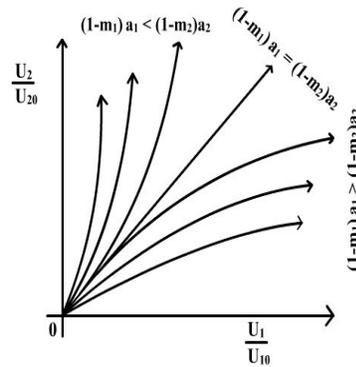


Fig.17

5 Conclusions

It is observed that the fully washed out state is unstable. Equilibrium state- I (Ammensal survives and Enemy is washed out state) turned out to be unstable and Equilibrium state-II (Enemy survives and Ammensal is washed out state) is conditionally stable. Also the authors noticed that Co-existence state is asymptotically stable.

6 Future Work

Ecological situations such as the following are for further investigation.

- a. One can examine an ecosystem consisting of a prey, a predator with unlimited resources and a enemy- Ammensal to the prey with various resources.
- b. One can study the above model with harvesting of one or both the species.
- c. One can enforce the concept of harvesting at variable rate for a four species ecosystem in various possible situations.

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