Distance-related Properties of Corona of Certain Graphs

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Abstract

A graph \(G\) is called a \(m\)-eccentric point graph if each point of \(G\) has exactly \(m \geq 1\) eccentric points. When \(m=1\), \(G\) is called a unique eccentric point (u.e.p.) graph. Using the notion of corona of graphs, we show that there exists a \(m\)-eccentric point graph for every \(m \geq 1\). Also, the eccentric graph \(G_e\) of a graph \(G\) is a graph with the same points as those of \(G\) and in which two points \(u\) and \(v\) are adjacent if and only if either \(u\) is an eccentric point of \(v\) or \(v\) is an eccentric point of \(u\) in \(G\). We obtain the structure of the eccentric graph of corona \(G \circ H\) of self-centered or non-self-centered u.e.p graph \(G\) with any other graph \(H\) and obtain its domination number.

Keywords: Domination, Eccentricity, Eccentric Graph.

1 Introduction

The notion of distance [2] in graphs has been studied in the context of many applications such as communication networks. The distance related parameter, known as eccentricity of a point in a graph and the associated notions of eccentric points, \(m\)-eccentric point graphs\cite{1, 3, 6} and in particular, unique eccentric point (u.e.p)graphs\cite{4}, have also been well investigated. Another kind of graph known as corona\cite{6} \(G \circ H\) of two graphs \(G\) and \(H\) has also been well studied. Also, the concept of eccentric graph \(G_e\) of a graph \(G\) was introduced in \cite{5}, based on the notion of distance among points in \(G\). Here we show, using the notion of corona of graphs, that there exists a \(m\)-eccentric point graph for every \(m \geq 1\). We also obtain the eccentric graph of corona \(G \circ H\) where \(H\) is any graph and \(G\) is either
self-centered u.e.p graph or non-self-centered u.e.p graph and obtain its domination number.

![Graphs](image)

**Fig. 1:** a) Graph $G$  b) Graph $G_e$

We recall here certain basic definitions {1, 3, 6} related to graphs. A graph $G = (V, E)$ consists of a finite non-empty set $V$ (also denoted by $V(G)$) whose elements are called points or vertices and another set $E$ (or $E(G)$) of unordered pairs of distinct elements of $V$, called edges. In a graph $G$, the distance $d_G(u,v)$ or $d(u,v)$, when $G$ is understood, between two points $u$ and $v$ is the length of the shortest path between $u$ and $v$. The eccentricity $e_G(u)$ or simply, $e(u)$ of a point $u$ in $G$ is defined as $e(u) = \max_{v \in V(G)} d(u,v)$. For two points $u$, $v$ in $G$, the point $v$ is an eccentric point of $u$ if $d(u,v) = e(u)$. We denote by $E(v)$, the set of all eccentric points of a point $v$ in $G$. A graph $G$ is called a m-eccentric point graph if $|E(u)|$, the number of elements of $E(u)$ equals $m$, for all $u$ in $V(G)$. When $m=1$, $G$ is called a unique eccentric point (u.e.p) graph. The radius $r(G)$ and the diameter $diam(G)$ of a graph $G$ are respectively defined as $r(G) = \min \{e(u) \mid \text{for all } u \in V(G)\}$ and $diam(G) = \max \{e(u) \mid \text{for all } u \in V(G)\}$. A graph $G$ is called a self-centered graph if $r(G) = diam(G)$.

The eccentric graph [5] $G_e$ of a graph $G$ is a graph with the same points as those of $G$ and in which two points $u$ and $v$ are adjacent if and only if either $u$ is an eccentric point of $v$ or $v$ is an eccentric point of $u$ in $G$. A graph $G$ and its eccentric graph $G_e$ are shown in Fig.1.
The corona \( \{6\} \) \( G^\circ H \) of two graphs \( G \) and \( H \) is a graph made of one copy of \( G \) with points \( v_1, v_2, v_3, \ldots, v_n, n \geq 1 \) and \( n \) copies of another graph \( H \) such that for every \( i, 1 \leq i \leq n \), the point \( v_i \) is joined with all the points of the \( i \)\(^{th}\) copy of \( H \).

We also need the following well-known notions. A complete graph \( K_n \) on \( n \) points, is a graph in which there is an edge between every pair of distinct points. The complement \( \overline{G} \) of a graph \( G \) is a graph having the same points as those of \( G \) and such that two points \( x \) and \( y \) are adjacent in \( G \) if and only if \( x \) and \( y \) are not adjacent in \( G \). The union \( G_1 \cup G_2 \) of two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) is the graph \( G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2) \) and the join \( G_1 + G_2 \) of \( G_1 \) and \( G_2 \) is a graph obtained from \( G_1 \cup G_2 \) by joining every point of \( G_1 \) with every other point of \( G_2 \). For three or more graphs \( G_1, G_2, G_3, \ldots, G_n \) the sequential join \( G_1 + G_2 + G_3 + \ldots + G_n \) is the graph \((G_1 + G_2) \cup (G_2 + G_3) \cup \ldots \cup (G_{n-1} + G_n)\). In a graph \( G=(V,E) \) a subset \( S \subseteq V \) is called a dominating set if each point \( u \) in \( V-S \) has a neighbour in \( S \) i.e. \( u \) is adjacent to some point in \( S \). The cardinality of a minimum dominating set of a graph \( G \) is called its domination number and it is denoted by \( \gamma(G) \).

2 Eccentric Point Graphs

In this section we make use of the notions of corona and \( u.e.p \) graphs to show there exists, for every \( m \geq 1 \), an \( m \)-eccentric point graph.

**Lemma 2.1** Let \( G \) be a graph whose eccentric points are \( p_1, p_2, p_3, \ldots, p_1 \), for some \( l \geq 1 \). Let \( H \) be any other graph. In the corona \( G^\circ H \), all the points of \( G^\circ H \) each of which is joined with \( p_i \), for some \( i \), \( 1 \leq i \leq 1 \), are the only eccentric points of \( G^\circ H \).

Proof. Let \( V(G) = \{ v_1, v_2, v_3, \ldots, v_{n-l}, p_1, p_2, p_3, \ldots, p_1 \} \) such that \( p_i \)'s are eccentric points of \( G \). Let \( H \) be any graph on \( m \) points. Let \( V(G^\circ H) = \{ v_i, p_k, u^i_j \mid 1 \leq i \leq n-1, 1 \leq k \leq l, 1 \leq t \leq n \) and \( 1 \leq j \leq m \} \) such that for a fixed \( i \), with \( 1 \leq i \leq n-l \), the points \( u^i_j \) for all \( 1 \leq j \leq m \), are joined with \( v_i \) while for a fixed \( i \), with \( n-l \leq i \leq n \), the points \( u^i_j \) for all \( 1 \leq j \leq m \), are joined with \( p_i \).

Then, let us prove that every point \( u^i_j \) for all \( n-l+1 \leq i \leq n \) and \( 1 \leq j \leq m \) is an eccentric point. Suppose that for some \( n-l+1 \leq i \leq n \) and \( 1 \leq j \leq m \), \( u^i_j \) is not an eccentric point. Then consider the point \( v \in V(G) \) whose eccentric point in \( G \) is \( p_i \) to which the point \( u^i_j \) is attached in \( G^\circ H \). Let for some \( n-l+1 \leq i \leq n \) and \( 1 \leq j \leq m \), \( u^i_j \) be the eccentric point of \( v \) in \( G^\circ H \). Then \( d_{G^\circ H}(v) = d_{G^\circ H}(v, u^i_j) = d_G(v, p_i) + 1 < d_G(v, p_i) + 1 = d_{G^\circ H}(v, u^i_j) \).
That is \( e_{G^*H}(v) < d_{G^*H}(v, u^i_j) \), which is a contradiction and hence every point \( u^i_j \), \( n-l+1 \leq i \leq n \) and \( 1 \leq j \leq m \) is an eccentric point. Now, it remains to prove that (1) no point of \( G \) as a point of \( G^*H \) is an eccentric point in \( G^*H \) and (2) no point of \( u^i_j \) for \( 1 \leq i \leq n-l \) and \( 1 \leq j \leq m \) is an eccentric point in \( G^*H \).

In order to prove (1), suppose that \( u \) is an eccentric point of \( G^*H \). Then there exists a point \( v \in G^*H \) for which \( u \) is the eccentric point. Then \( u \) cannot be any \( v_i \), \( 1 \leq i \leq n-l \) or any \( p_i \), \( n-l+1 \leq i \leq n \) for otherwise \( e_{G^*H}(v) = d_{G^*H}(u, v) < d_{G^*H}(u, v) + 1 = d_{G^*H}(v, u^i_j) \) which is a contradiction, due to the fact that any path between \( v \) and \( u^i_j \) passes through either \( v_i \) or \( p_i \). Thus no point of \( G \) as a point of \( G^*H \), can be an eccentric point of \( G^*H \).

For proving (2), suppose that \( u^i_j \) for some \( 1 \leq i \leq n-l \), \( 1 \leq j \leq m \) is an eccentric point of some point \( v \) in \( G^*H \), then \( e_{G^*H}(v) = d_{G^*H}(v, u^i_j) = d_{G^*H}(v, v_i) + 1 < d_{G^*H}(v, u^i_j) \). That is \( e_{G^*H}(v) < d_{G^*H}(v, u^i_j) \) for some \( n-l+1 \leq k \leq n \), which is a contradiction.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{a) Graph G  \hspace{1cm} b) Graph H \hspace{1cm} c) Corona G^*H}
\end{figure}

**Remark 2.2.** With the graphs \( G \) and \( H \) as shown in Fig. 2, the eccentric points of \( G^*H \) are \( u^1_1, u^2_1, u^1_4, u^2_4 \). It can be noticed that no point \( v_i \), \( 1 \leq i \leq 4 \), of \( G \)
is an eccentric point of \( G^oH \) and all the points of \( G^oH \) that are joined with the points that are eccentric points of \( G \) are the only eccentric points of \( G^oH \).

**Theorem 2.3.** Let \( G \) be a u.e.p graph on \( n \) points and \( H \) be any graph on \( m \) points. Then the corona of \( G \) and \( H \), \( G^oH \) is a \( m \)-eccentric point graph.

Proof. Let \( G \) be a u.e.p graph with \( n \) points \( v_1, v_2, v_3, ..., v_n \) and \( H \) be any graph on \( m \) points. Let the points of \( G^oH \) that are points in the \( k \)th copy of \( H \), be \( u_j^k, 1 \leq j \leq m \). For any two points \( v_i, v_k \) of \( G \) such that \( v_k \) is the only eccentric point in \( G \) of \( v_i \) by Lemma 2.1, the points \( u_j^k, 1 \leq k \leq m \), are the eccentric points in \( G^oH \) of \( v_i \) as well as \( u_j^k, 1 \leq j \leq m \). No other point \( u_j^r \) for some \( r \neq k \), \( 1 \leq r \leq n \), can be an eccentric point in \( G^oH \) of \( v_i \) or \( u_j^r, 1 \leq j \leq m \), since \( d_{G^oH}(v_i, u_j^r) = d_G(v_i, v_k) + 1 < d_G(v_i, v_k) + 1 = d_{G^oH}(v_i, u_j^k) \) and \( d_{G^oH}(u_j^i, u_j^r) = d_{G^oH}(u_j^i, u_j^r) + 1 < d_{G^oH}(v_i, u_j^k) + 1 = d_{G^oH}(u_j^i, u_j^r) \). This implies that \( E(v_i) = \{ v_k^1, v_k^2, v_k^3, ..., v_k^m \} \). If \( v_k \) is an eccentric point of \( v_i \) in \( G \) and \( E(u_j^r) = \{ u_j^1, u_j^2, u_j^3, ..., u_j^m \} \) if \( v_q \) is an eccentric point of \( v_p \) in \( G \). Therefore, \( |E(u)| = m \), for all points \( u \) in \( G^oH \) and so \( G^oH \) is a \( m \)-eccentric point graph.

As a consequence of the Theorem 2.3, we obtain the following corollary.

**Corollary 2.4.** For every \( m \geq 1 \), there exists a \( m \)-eccentric point graph.

## 3 Eccentric Graph of Corona of u.e.p Graph with any other Graph

In this section we obtain the eccentric graph of corona of a u.e.p graph with any other graph.

**Theorem 3.1.** Let \( G \) be a self-centered u.e.p graph on \( 2n \) points and \( H \) be a graph on \( m \) points. Then the eccentric graph \( (G^oH)_e \) is the union of \( n \) copies of \( K_1 + \overline{K_m} + \overline{K_m} + K_1 \).

Proof. Let \( G \) be a self-centered u.e.p graph on \( 2n \) points and \( H \) be a graph on \( m \) points. Let \( V(G) = \{ v_1, v_2, v_3, ..., v_{2n} \} \) such that \( v_i \) and \( v_{i+n} \) \( (1 \leq i \leq n) \) are eccentric points of each other, in the graph \( G \). Then by Lemma 2.1, all the points \( u_j^i \) \( (1 \leq i \leq 2n; 1 \leq j \leq m) \) are eccentric points in \( G^oH \) because all the points of \( G \) are eccentric points in \( G \). This implies that the eccentric points of \( u_j^i \) and \( v_i \) are \( u_j^{i+n} \) \( (1 \leq i \leq n, 1 \leq j \leq m) \) and the eccentric points of \( u_j^{i+n} \) and \( v_{i+n} \) are \( u_j^i \) \( (1 \leq i \leq n, 1 \leq j \leq m) \). Now in \( (G^oH)_e \), which has the same point set as \( G^oH \), the point \( v_i \) is adjacent with all the points \( u_j^{i+n} \), each of the points \( u_j^{i+n} \) is adjacent with every
point $u_i^j$ and all the points $u_i^j$ are adjacent with $v_{i+n}$ ($1 \leq i \leq n$, $1 \leq j \leq m$).

Therefore, $(G^eH)_e$ is the union of $n$ copies of $K_1 + \overline{K_m} + \overline{K_m} + K_1$.

**Theorem 3.2.** Let $H$ be a graph on $m$ points and $G$ be a non-self-centered u.e.p graph on $n$ points having the properties (i) $P(G) = EP(G)$, (ii) $|P(G)| = 2t$, $t > 1$, (iii) for every $u$ in $P(G)$ there is at least one $v$ in $V(G) - P(G)$ such that $E(v) = \{u\}$, then $(G^eH)_e$ is a union of $t$ copies of $\overline{K_{t_1}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_1}}$, for some $t_1 \geq 1$ and $t_j \geq 1$, $t_i$ and $t_j$ depending on $G$ and $H$.

Proof. Let $H$ be a graph on $m$ points and $G$ be a non-self-centered u.e.p graph on $n$ points having the properties (i) $P(G) = EP(G)$, (ii) $|P(G)| = 2t$, $t > 1$, (iii) for every $u$ in $P(G)$ there is at least one $v$ in $V(G) - P(G)$ such that $E(v) = \{u\}$.

Let $V(G) = \{v_1, v_2, v_3, ..., v_n\}$. Let $v_1, v_2, v_3, ..., v_{2t}$ for some $t \geq 1$ be the peripheral vertices of $G$, so that $|P(G)| = 2t$. For $1 \leq i \leq t$, let $v_i$ and $v_{i+t}$ be the eccentric points of each other. Let $V(G^eH) = V(G) \cup \{u_i^j, 1 \leq j \leq m, 1 \leq i \leq n\}$.

Then by Lemma 2.1, all the points $u_i^j$, $1 \leq i \leq 2t$, $1 \leq j \leq m$ are the eccentric points of $G^eH$ because $v_1, v_2, v_3, ..., v_{2t}$ are the eccentric points in $G$. This implies that for $1 \leq i \leq t$, $1 \leq j \leq m$, $u_i^{j+1}$ is the eccentric point of $u_i^j$, $v_i$, $v_k$ as well as $u_j^{k+1}$ with $E(v_k) = \{v_i\}$ for $v_k \in V(G)$. Also, $1 \leq i \leq t$, $1 \leq j \leq m$, $u_i^j$ is the eccentric point of $u_i^{j+1}$, $v_{i+t}$, $v_k$ as well as $u_j^{k+1}$. Since the eccentric graph $G_e$, of any graph $G$, is constructed with the same points as those of $G$ and each edge of $G_e$ joins a point $x$ with the eccentric points of $x$ treated as a point of $G$. Thus, the structure of $(G^eH)_e$ is clearly, union of $t$ copies of $\overline{K_{t_1}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_1}}$, for some $t_1 \geq 1$ and $t_j \geq 1$. Note that $t_i$ and $t_j$ depend on $G$ and $H$.

**Example 3.3.** A non-self-centered u.e.p graph $G$ and $H$ on $m=2$ points are shown in Fig.3. It is clear that in $G$, the eccentric points are $v_1$, $v_2$, $v_3$ and $v_4$ and $E(v_1) = E(v_{10}) = E(v_{12}) = \{v_1\}$; $E(v_1) = E(v_5) = E(v_7) = \{v_4\}$;
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Fig. 3: a) A non-self centered u.e.p Graph G  b) Graph H

E(v_2) = E(v_6) = E(v_8) = \{v_3\} ; E(v_3) = E(v_{11}) = \{v_2\}. The corona of G and H is shown in Fig.4. Note that v_1^1, v_2^1, v_1^2, v_2^3, v_1^3, v_2^4 are the eccentric vertices of (G°H)_e. The eccentric graph, (G°H)_e is union of 2 copies of \(K_{t_i} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}\), where m=2; t_i=7 and t_j=7 and it is shown in the Fig.5.

**Theorem 3.4.** Let G be a self-centered u.e.p graph on 2n points and H be a graph on m points. Then the domination number \(\gamma(G°H)_e = 2n\).

Proof. Let G be self-centered u.e.p graph on 2n points and H be a graph on m points. Now, by Theorem 3.1, (G°H)_e is union of n copies of \(K_{t_i} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}\). In each copy, there are two v_i's dominating the remaining points in that copy. Therefore, \(\gamma(G°H)_e = 2n\).

**Theorem 3.5.** Let H be a graph on m points and G be a non-self-centered u.e.p graph on n points having the properties (i) P(G) =EP(G), (ii) \(|P(G)| =2t, t>1\), (iii) for every u in P(G) there is at least one v in V(G) – P(G) such that E(v) =\{u\}, then the domination number \(\gamma(G°H)_e = 2t\).

Proof. Let H be a graph on m points and G be a non-self-centered u.e.p graph on n points having the properties (i) P(G) =EP(G), (ii) \(|P(G)| =2t, t>1\), (iii) for every u in P(G) there is at least one v in V(G) – P(G) such that E(v) =\{u\}. Then by

**Theorem 3.2.** (G°H)_e is union of t copies of \(K_{t_i} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}\), for
some \(t_i \geq 1\) and \(t_j \geq 1\), \(t_i\) and \(t_j\) depending on \(G\) and \(H\). In each copy, there are two points dominating the remaining points in that copy. Therefore, \(\gamma(G \circ H) = 2t\).

Fig. 4: Corona of \(G\) and \(H\), \(G \circ H\)

4 Conclusion

The structure of eccentric graph of \(m\)-eccentric point graph can be investigated. Also the problem of finding a graph whose eccentric graph is a \(m\)-eccentric point graph remains open.
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