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Wiener Dimension of Certain Trees

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Abstract

The closeness or the distance of a vertex u in a graph G, denoted by $\delta_G(u)$, is the sum of distances between u and all other vertices of G. The Wiener dimension of a connected graph is defined as the number of different distances of its vertices. In this paper we prove that any tree has Wiener dimension 2 if and only if it is isomorphic to a star graph or a bi-star graph. We also identify certain classes of trees with Wiener dimension 3 and 4.

Keywords: *Bi-star, comb, Star, trees, Wiener dimension.*

1 Introduction

Wiener index was introduced by the American chemist Harold Wiener in 1947 [1]. In the mathematical field of graph theory, the distance between two vertices u and v of a graph G is the number of edges in a shortest path between u and v and is denoted by $d_G(u,v)$. The definition of the Wiener index in terms of distances between vertices of a graph was first given by Hosoya [2]. Wiener index of a graph G is defined as the sum of distances between all pairs of vertices in G: $W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)$. For a vertex u in graph G, the distance or closeness of u is defined as $\delta_G(u) = \sum_{v\in V(G)} d_G(u,v)$. Suppose that $\{\delta_G(u) \mid u \in V(G)\} = \{\delta_1, \delta_2, \ldots, \delta_k\}$ and G contains t_i vertices of distance d_i, $1 \le i \le k$, then the Wiener index of G can be expressed as $W(G) = \frac{1}{2}\sum_{i=1}^{k} t_i d_i$ and the Wiener dimension $dim_W(G)$ is defined as k. Alizadeha et al. [3] have studied Wiener dimension of (5,0)-Nanotubical Fullerenes. Wiener index is extensively studied in Chemistry [4, 5] and Mathematics [1, 6, 7].

For the graph G in Fig. 1, $\delta_G(v_1) = 3$, $\delta_G(v_2) = 4$, $\delta_G(v_3) = 4$, $\delta_G(v_4) = 5$, $W(G) = 8$ and $dim_W(G) = 3$.

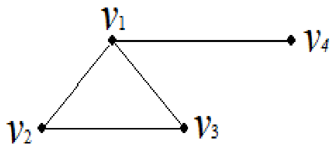


Fig. 1: Graph G with $dim_W(G) = 3$

Trees are the first special classes of graphs that are studied extensively by many authors [6, 8] and Dobrynin et al. [9] give a detailed survey on the Wiener index of trees. In addition, Gutman and Skrekovski [10] proved that for a connected graph G, the Wiener index is related to the betweenness centrality $B(v)$ of the vertices $v \in V(G)$, a quantity used in the theory of social networks, which measures the number of times a vertex lies on a shortest path between two other vertices. There have been a series of research articles over Wiener index of trees, for instance see [6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19]. In this paper we introduce a technique to compute $\delta_G(v)$ for $v \in V(G)$ without using distances. Further, we prove that a tree has Wiener dimension 2 if and only if it is isomorphic to a star graph or a bi-star graph.

2 Basic Definitions

Definition 2.1 [3] *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The distance $d_G(u,v)$ between two vertices $u, v \in V(G)$ is the minimum number of edges on a path in G between u and v.*

Definition 2.2 [16, 20] *Let G be a graph. The closeness or the distance of a vertex u in G, denoted by $\delta_G(u)$, is defined as $\delta_G(u) = \sum_{v \in V(G)} d(u,v)$.*

Thus, one can also define the Wiener index in a slightly different way:

$W(G) = \frac{1}{2} \sum_{u \in V(G)} \delta_G(u)$ where $\frac{1}{2}$ compensates for the fact that each path between u and v is counted in $\delta_G(u)$ as well as in $\delta_G(v)$. When there is no ambiguity, we denote $\delta_G(u)$ as $\delta(u)$.

Definition 2.3 [20] *The set of vertices of a graph G that minimizes the closeness of vertices is called the median set of G.*

Definition 2.4 [16] *Let G be a graph. The diameter of G, denoted by $diam(G)$ is defined as $diam(G) = max_{u,v \in V(G)} d_G(u,v)$, where maximum is taken over all pairs of vertices in G.*

Definition 2.5 [21] *A tree is an undirected acyclic graph in which any two vertices are connected by exactly one path.*

Definition 2.6 *Let T be a tree. A vertex $v \in V(T)$ is called branching point of T, if $deg_T(v) \geq 3$. If $deg_T(v) = 1$, the vertex v is named a pendent vertex or a leaf of T.*

Definition 2.7 *A star $K_{1,k}$ is a complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = 1$ and $|V_2| = k$. The vertex of degree k is called the central vertex and all other vertices are called leaves.*

Definition 2.8 [22] *A tree T is called a caterpillar if the tree obtained from T by removing all pendent vertices induce a path. The path that is formed by the non-leaves is known as the spine of the caterpillar.*

Definition 2.9 *Comb is a graph obtained by joining a single pendent edge to each vertex of a path. The path is called the spine of the comb. A comb with m spine vertices is denoted by $C(m)$.*

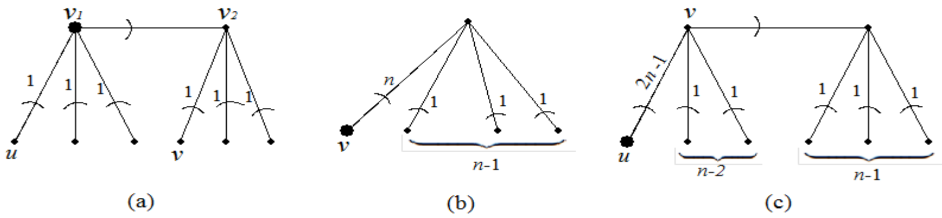
3 Trees with Wiener dimension 2

Among all the trees on n vertices, the star $K_{1,n-1}$ has the lowest Wiener index and the path P_n has the largest Wiener index [23] and hence for any tree T on n vertices, $W(K_{1,n-1}) \leq W(T) \leq W(P_n)$ [9]. In this paper we develop a new technique to compute $\delta(v)$ for $v \in V(G)$ without finding the actual distance from v to any other vertex in G and formulate it as T-Closeness Lemma. In fact the Partition Technique [24] is modified to compute $\delta(v)$.

Definition 3.1 *The bi-star graph is the graph obtained from two disjoint copies of $K_{1,m}$ and $K_{1,n}$ by joining the central vertices by an edge and is denoted by $B_{m,n}$.*

Fig. 2: (a) u is a child of v_1 and v is a child of v_2; (b) v is a pendent vertex in $K_{1,n}$; (c) v is a spine vertex and u is a pendent vertex in $B_{n-1,n-1}$

Definition 3.2 *Let $v \in V(T)$. For $e \in E(T)$, define congestion on e with respect to v denoted by $c_v(e)$ as the number of times e is crossed while traversing from v to every other vertex of T. See Fig. 2.*



Theorem 3.3 *Let* $v \in V(T)$. *Then* $\delta_T(v) = \sum_{e \in E(T)} c_v(e)$.

Proof. Any path P of length $d_T(v, w)$ from the vertex v to a vertex w in T contributes congestion 1 on each of the edges in P. This is true for all paths from v to every other vertex of T. This implies $\delta_T(v) = \sum_{w \in V(T)} d(v, w) = \sum_{e \in E(T)} c_v(e)$.

Lemma 3.4 (T*-Closeness Lemma*) *Let* T *be a tree and* $v \in V(T)$. *For every edge* e *in* T, *let* T_e *be the component of* $T - e$ *which does not contain* v. *Then* $\delta(v) = \sum_{e \in E(T)} |V(T_e)|$.

Proof. Every edge e of T is a cut edge whose removal disconnects T into two subtrees T_e and $T_e^{'}$, one of which contains v, say $T_e^{'}$. Then all paths from v to every vertex of T_e yield $|V(T_e)|$ as the congestion $c_v(e)$ on e. By Theorem 3.3, $\delta(v) = \sum_{e \in E(T)} c_v(e) = \sum_{e \in E(T)} |V(T_e)|$.

Definition 3.5 *Let* v *be a cut vertex of* G. *The* v*-components of* G *are subgraphs induced by the components of* $G - v$ *together with* v.

Lemma 3.6 *If* T *is a tree of order* $n > 4$ *and if* T *contains a path of length at least* 3 *all of whose internal vertices have degree* 2, *then* $dim_W(T) \geq 3$.

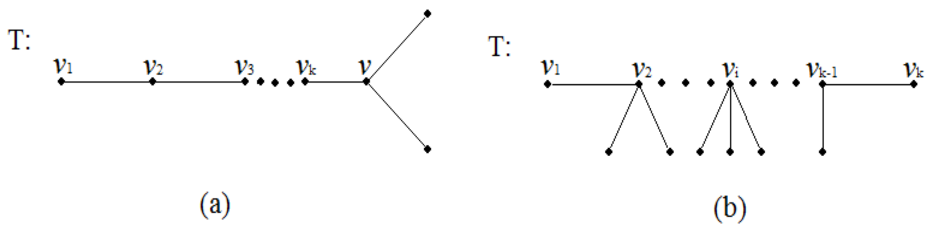


Fig. 3: (a) Path $P: v_1 v_2 v_3 \ldots v_k v$ as a v-component of $T - v$
(b) Caterpillar with k spine vertices

Proof. Let T be a tree on n vertices, $n > 4$ and let $v \in V(T)$. Let path $P: v_1 v_2 v_3 \ldots v_k v$, $k \geq 2$, be a v-component of $T - v$. For a vertex x in T with degree 2, let T_x denote the subtree of T rooted at x and containing the vertex v.

Now, $\delta_T(v_1)=(n-1)+\delta_{T_{v_2}}(v_2)=(n-1)+(n-2)+\delta_{T_{v_3}}(v_3)$;
$\delta_T(v_2)=1+\delta_{T_{v_2}}(v_2)=1+(n-2)+\delta_{T_{v_3}}(v_3)$ and $\delta_T(v_3)=3+\delta_{T_{v_3}}(v_3)$. Hence $\delta_T(v_1)=\delta_T(v_2)$ implies $n=2$; $\delta_T(v_2)=\delta_T(v_3)$ implies $n=4$ and $\delta_T(v_3)=\delta_T(v_1)$ implies $n=3$, which is a contradiction since $n>4$. Hence $\delta_T(v_1)\neq\delta_T(v_2)\neq\delta_T(v_3)$. The case when $v_3=v$ is not ruled out. Thus $dim_W(T)\geq 3$. See Fig. $3(a)$.

Lemma 3.7 *If T is a tree with $dim_W(T)=2$, then T is a caterpillar.*

Proof. Let P be a longest path in T. Then the end vertices of P are pendent vertices in T. If not, the longest path property of P will be violated. Let $P: v_1v_2\ldots v_k$, $k\geq 2$. Since $dim_W(T)=2$, by Lemma 3.6, a subtree of T rooted at v_i and not containing $v_1,v_2,\ldots,v_{i-1},v_{i+1},\ldots,v_k$ is isomorphic to a star graph rooted at v_i, $2\leq i\leq k-1$. See Fig. $3(b)$. This implies that T is a caterpillar.

Lemma 3.8 *Let T be a caterpillar with k spine vertices and $dim_W(T)=2$. Then $k\leq 2$.*

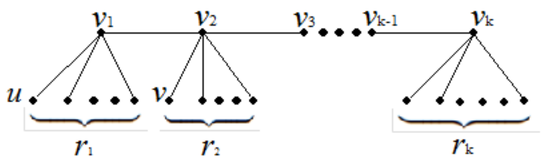


Fig. 4: Caterpillar with k spine vertices

Proof. Suppose not. Let $v_1,v_2,\ldots,v_k$, $k\geq 3$ be the spine vertices from left to right. Let the number of pendent edges incident at v_i be r_i, $r_i\geq 0$, $1\leq i\leq k$. See Fig. 4. Let u be a child of v_1 and v be a child of v_2.
By T-Closeness Lemma,

$$\delta(u)=(n-1)+(r_1-1)+(n-(r_1+1))+\delta_{T_{v_2}}(v_2)=(2n-3)+\delta_{T_{v_2}}(v_2)=(2n-3)+r_2+(n-(r_1+r_2+2))+\delta_{T_{v_3}}(v_3)=3n-r_1-5+\delta_{T_{v_3}}(v_3).$$

On the other hand

$$\delta(v)=(n-1)+(r_2-1)+(r_1+1)+r_1+(n-(r_1+r_2+2))+\delta_{T_{v_3}}(v_3)=2n-3+\delta_{T_{v_3}}(v_3).$$

$\delta(u)=\delta(v)\Rightarrow n=r_1+2\Rightarrow T$ is a star graph with $k=1$, a contradiction. Hence $\delta(u)\neq\delta(v)$. Now

$$\delta(v_1)=r_1+(n-(r_1+1))+r_2+(n-(r_1+r_2+2))+\delta_{T_{v_3}}(v_3)=2n-r_1-3+\delta_{T_{v_3}}(v_3).$$

Therefore $\delta(v)=\delta(v_1)\Rightarrow r_1=0$, a contradiction.

Similarly $\delta(u)=\delta(v_1)\Rightarrow 3n-5=2n-3\Rightarrow n=2$, a contradiction.
$\delta(u)\neq\delta(v)\neq\delta(v_1)$, a contradiction to $dim_W(T)=2$. Hence $k\leq 2$.

Lemma 3.9 *Let T be either the star graph $K_{1,n}$ or the bi-star graph $B_{n,n}$, $n>1$. Then $dim_W(T)=2$.*

Proof. **Case** 1**:** Let $G\cong K_{1,n}$. Then $c(v)=deg(v)=n$ if v is the central vertex of the graph and $c(v)=2n-1$ when v is a pendent vertex. Therefore $dim_W(G)=2$. See Fig. $2(b)$.

Case 2 **:** Let G be a bi-star graph. When v is a spine vertex, $c(v)=(n+1)+2n=3n+1$ and when u is a pendent vertex, $c(u)=(n+1)+(2n-1)+(2n-1)=5n-1$. See Fig. $2(c)$. This implies $dim_W(T)=2$.

Theorem 3.10 *Let T be a caterpillar with spine length k and $dim_W(T)=2$. Then T is isomorphic to $K_{1,n}$ or $B_{n,n}$.*

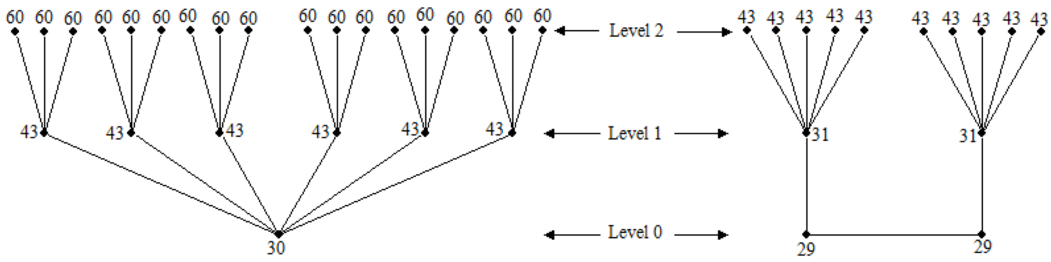
Proof. By Lemma 3.8, $k\leq 2$. If $k=1$, then T is isomorphic to $K_{1,n}$. If $k=2$, then T is isomorphic to a bistar with spine (v_1,v_2). If v_1 has r_1 leaves adjacent to it and v_2 has r_2 leaves adjacent to it, then

$$\delta(x)=\begin{cases}1+2r_1+3r_2 & \text{if } x \text{ is a leaf adjacent to } v_1\\ 1+2r_2+3r_1 & \text{if } x \text{ is a leaf adjacent to } v_2\\ r_1+1+2r_2 & \text{if } x=v_1\\ r_2+1+2r_1 & \text{if } x=v_2\end{cases}$$

Hence $dim_W(T)=2$ only if $r_1=r_2=n$ and T is isomorphic to $B_{n,n}$.

Lemma 3.9 and Theorem 3.10 imply the following characterization of trees with Wiener dimension 2.

Theorem 3.11 *Let T be a tree. Then $dim_W(T)=2$ if and only if it is a star graph $K_{1,n}$ or a bi-star graph $B_{n,n}$, $n>1$.*



(a) $dim_W(T_{6,3}) = 3$ (b) $dim_W(F_{2,6}) = 3$

Fig. 5: (a) Crystal tree; (b) Firecracker graph

4 Trees with Wiener dimension 3 and 4

Definition 4.1 *A crystal tree is defined as a tree in which every leaf member of* $K_{1,n}$ *is identified or merged with the root of a copy of* $K_{1,r}$. *Crystal tree is denoted by* $T_{n,r}$. *The root node is said to be in level* 0. *Vertices at distance* i *from the root node are said to be at level* i, $1 \le i \le 2$.

Theorem 4.2 *Let* $T_{n,r}$ *be a crystal tree. Then* $dim_W(T_{n,r}) = 3$. ***Proof.*** Let $T_{n,r}$ be a crystal tree with $n = 1, r \ge 2$. $T(u)$ for vertex u in level i is different from $T(u)$ for vertex u in level j, $i \ne j$. On the other hand $T(u)$ is the same for every u in a level i, $1 \le i \le 2$. See Fig. $5(a)$. Hence $dim_W(T_{n,r}) = 3$.

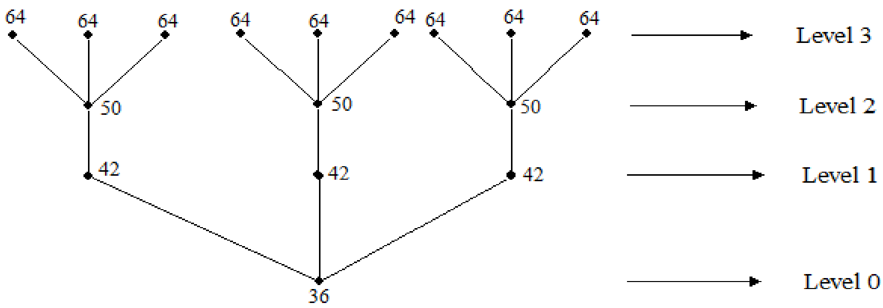


Fig. 6: Banana tree $B_{3,4}$ with $dim_W(B_{3,4}) = 4$

Definition 4.3 [25] *An* (n,r) *-firecracker is a graph obtained by the concatenation of* n *number of* r *-stars by linking one leaf from each. Firecracker graph is denoted by* $F_{n,r}$.

The proof of Theorem 4.4 and Theorem 4.6 is similar to Theorem 4.2 .

Theorem 4.4 *Let* $F_{2,r}$ *be a firecracker graph. Then* $dim_W(F_{2,r}) = 3$.

See Fig. $5(b)$.

Definition 4.5 [26] *An* (n,k) *banana tree is a graph obtained by connecting one leaf of each of* n *copies of a* k*-star graph with a single root vertex that is distinct from all the stars. Banana tree is denoted by* $B_{n,k}$.

Theorem 4.6 *Let* T *be a banana tree* $B_{n,k}$. *Then* $dim_W(B_{n,k}) = 4$.

See Fig. 6.

5 Conclusion

The technique developed in this paper is a powerful tool to compute Wiener dimension of trees. We have characterized certain trees with Wiener dimension 2. It is an interesting line of research to characterize trees with Wiener dimension k, $k \geq 3$. In this direction, using T-Closeness Lemma we formulate a conjecture on the Wiener dimension of a comb graph.

Conjecture. Let $C(m)$ be a comb graph, $m \geq 2$.

Then $$dim_W(C(m)) = \begin{cases} m-1 & when\ m \equiv 1,5,9 (mod\ 16) \\ m & when\ m\ is\ even\ or\ m \equiv 13 (mod\ 16) \\ m+1 & when\ m \equiv 3 (mod\ 4)\ or\ m \equiv 3 (mod\ 16) \end{cases}$$

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