

Fractional Order C^1 Cubic spline

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Abstract

In the field of computer graphics, spline curves and surfaces are playing a vital role. In fact it is known as standard tool for computer graphics. Due to this very reason, lots of work has been done in this field and still going on. This research adopted a novel technique called Caputo fractional derivatives to find all unknowns that appear in spline cubic polynomial. This new method of finding unknowns could be an important technique in the areas where one does not need a curve to be C^2 . Fractional derivative technique can further be applied on other kind of spline curves. Our technique provides an alternate approach to develop piecewise cubic spline polynomials. These polynomials are C^1 continuous in nature.

Keywords: *Fractional Order derivative, Cubic Spline, Continuity*

1 Introduction

The field of Computer-Aided Geometric Design was first developed to establish and improve industrial advantages pertaining to aircrafts, automotives and assembling ships and is now disseminating information towards all walks of life including animations and pharmaceutical designs. Visualization on a computer is the first step towards of the actual product. From there, we can easily pinpoint what would be the pitfalls in the end product so that we can modify. It is a continuous product until a refined form is achieved. Next step in the production is to create tools and dies using geometry stored in the computer.

Curve and surfaces are the basis of Computer-Aided Geometric Design. Intricate results can be developed and refined by using CAGD tools. Although, large numbers of curves and surface segments are combined together to practically imply the ideas [6], however, the individual segments of the curve also plays are imperative in the field of CAGD. Spline curves and surfaces are classical tools for geometric modeling in computer Aided Geometric Designing (CAGD) and Computer Graphics (CG). Different spline functions have been developed such as Natural Spline, Beta Spline, B-Spline [5], Trigonometric Spline, and Rational Spline for curves and surfaces in Computer Aided Geometric Design (CAGD), Computer Graphics (CG), Numerical Analysis, Computer Aided Design and Geometric Modeling (Farin, 2002). Number of researchers discussed cubic Spline solutions for Initial value problems (IVP) and boundary value problems (BVP) of fractional differential equations [7-10]. However in this paper Caputo fractional order derivatives have been used for CAGD.

For a sufficiently well-behaved function $f(t)$ where $t \in \mathbb{R}^+$ can be defined as the derivative of a positive non-integer order in two different senses, that are referred as to Caputo (C) derivative [1, 2, 4]. The fractional order derivative of $f(x) \in AC^n[a, \infty)$, order α with $n-1 < \alpha \leq n$, is defined as:

$$({}^C D_a^\alpha f)(\xi) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\xi)}{(x-\xi)^{\alpha-n+1}} d\xi, \quad x > a. \quad (1)$$

1.1

Where;

$$\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du \quad (2)$$

This paper focuses on three different aspects including the inculcation of an alternative method for computer aided geometric design using piecewise cubic spline polynomials based on Caputo Fractional order derivative. Secondly, the paper depicts effective results for smooth piecewise curves passing through the given data points. Thirdly, a comparison is generated with the existing ordinary piecewise cubic spline methods and the novel technique has been discussed with the help of different examples.

2 Fractional Order Cubic Spline

Since spline is a standard tool to computer graphic and lot of work have been done in this area. Researchers are working on different spline curves with novel ideas. This research is also an effort to generate spline curve with an innovative idea of fractional derivatives. When we talk about cubic spline curves that means there are polynomials of degree three as defined blow [3]:

$$S_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \text{ for } x \in [x_i, x_{i+1}] \quad (3)$$

and

$$S_{i+1}(x) = a_{i+1}(x - x_{i+1})^3 + b_{i+1}(x - x_{i+1})^2 + c_{i+1}(x - x_{i+1}) + d_{i+1} \quad (4)$$

$$\text{for } x \in [x_{i+1}, x_{i+2}] \quad (5)$$

These two polynomials are joining three points namely (x_i, y_i) , (x_{i+1}, y_{i+1}) and (x_{i+2}, y_{i+2}) with following properties:

$$\left. \begin{aligned} S_i(x_i) = y_i, S_{i+1}(x_{i+2}) = y_{i+2}, S_i(x_{i+1}) = S_{i+1}(x_{i+1}) \\ S_{i+1}(x_{i+1}) = y_{i+1}, \\ S_i^\alpha(x_{i+1}) = S_{i+1}^\alpha(x_{i+1}), \\ S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) \end{aligned} \right\} \quad (6)$$

Where " α " is fractional order derivatives and $0 < \alpha < 1$. We can generate a C^1 continuous cubic spline curve. The above continuity and differentiability conditions show that, there are eight unknowns namely, $a_i, b_i, c_i, d_i, a_{i+1}, b_{i+1}, c_{i+1}$ and d_{i+1} . Eight unknowns and five conditions are given above. We can assign three different values to α between 0 and 1, in order to get eight linear equations to solve for a unique solution for unknowns constants. In order to find fractional order derivatives of $S_i(x)$ and $S_{i+1}(x)$, we need to find first order derivatives.

$$S_i'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i, \text{ for } x \in [x_i, x_{i+1}] \quad (7)$$

$$S_{i+1}'(x) = 3a_{i+1}(x - x_{i+1})^2 + 2b_{i+1}(x - x_{i+1}) + c_{i+1}, \text{ for } x \in [x_{i+1}, x_{i+2}] \quad (8)$$

$$\text{Left-fractional} = \frac{1}{\Gamma(1-\alpha)} \int_t^{x_{i+1}} \frac{3a_i(\tau-x_i)^2 + 2b_i(\tau-x_i) + c_i}{(x_{i+1}-\tau)^\alpha} d\tau \quad (9)$$

$$\text{Right-fractional} = \frac{1}{\Gamma(1-\alpha)} \int_{x_{i+1}}^s \frac{3a_{i+1}(\tau-x_{i+1})^2 + 2b_{i+1}(\tau-x_{i+1}) + c_{i+1}}{(\tau-x_{i+1})^\alpha} d\tau \quad (10)$$

At point $x = x_{i+1}$, by applying the continuity and differentiable conditions 1.5 and varying the values of $\alpha \in (0,1)$ we have: the following linear system of equations form:

$A_{ij}B_i = C_i$, $i, j = 0,1, \dots, 5$. A_{ij} is matrix of coefficients whose values are as under:

$$\begin{aligned} A_{00} = 3h_i^2, \quad A_{01} = 2h_i, \quad A_{02} = 1, \quad A_{03} = 0, \quad A_{04} = 0, \quad A_{05} = 1, \quad A_{06} = 0, \\ A_{16} = y_{i+1} - y_i, \quad A_{56} = y_{i+2} - y_{i+1}, \quad A_{10} = h_i^3, \quad A_{11} = h_i^2, \quad A_{12} = h_i, \\ A_{13} = 0, \quad A_{14} = 0, \quad A_{15} = 0, \quad A_{50} = 0, \end{aligned}$$

$$\begin{aligned}
 A_{51} &= 0, \quad A_{52} = 0, \quad A_{53} = h_{i+1}^3, \quad A_{54} = h_{i+1}^2, \quad A_{55} = h_{i+1}, \\
 A_{j0} &= \{3(x_{i+1} - t)^{1-p_j}(2x_{i+1}^2 + 2x_i x_{i+1}(p_j - 3)) + x_i^2(p_j - 3)(p_j - 2) - (2t(x_{i+1} + \\
 &\quad x_i(p_j - 3))(p_j - 1) + t^2(p_j - 2)(p_j - 1)\}/(p_j - 3)(p_j - 2)(p_j - 1) \\
 A_{j1} &= 2(x_{i+1} - t)^{1-p_j}(-x_{i+1} - x_i(p_j - 2) + t(p_j - 1))/(p_j - 2)(p_j - 1) \\
 A_{j1} &= 2(x_{i+1} - t)^{1-p_j}(-x_{i+1} - x_i(p_j - 2) + t(p_j - 1))/(p_j - 2)(p_j - 1) \\
 A_{j2} &= (x_{i+1} - t)^{1-p_j}/(1 - p_j), \quad A_{j3} = 3(s - x_{i+1})^{3-p_j}/(3 - p_j) \\
 A_{j5} &= (s - x_{i+1})^{1-p_j}/(1 - p_j), \quad A_{j6} = 0
 \end{aligned} \tag{11}$$

For $j = 2, 3$ and 4 . Here we took $p_2 = \alpha, p_3 = \beta$ and $p_4 = \gamma$

There are six linear equations in eight unknown constants. Two constants can be calculated by using continuity conditions: $S_i(x_i) = y_i \Rightarrow d_i = y_i,$
 $S_{i+1}(x_{i+1}) = y_{i+1} \Rightarrow d_{i+1} = y_{i+1},$ (12)

Here s and t introduced in formula above will work as shape parameters.

Table 1: Points in xy-plane

X	0	3	6
Y	1	9	1

Table 2: Fractional order derivatives for α

α	0.90	0.92	0.95
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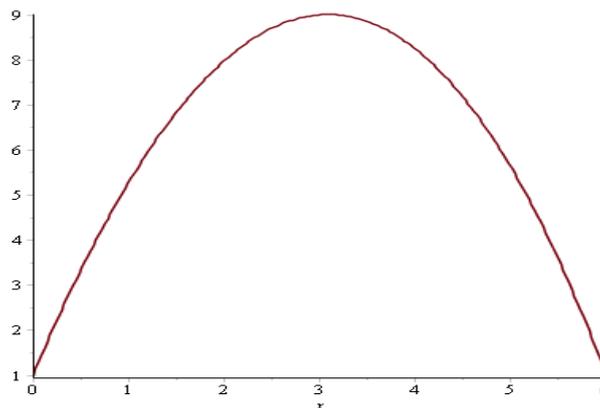


Fig.1 Fractional cubic spline using first derivative and fractional derivatives less than 1.

3 Comparison Ordinary Cubic Spline and Fractional Cubic Spline

It is evident from Fig. 2 that both the ordinary technique to find cubic spline curve and the technique which is introduced in this paper are comparable. We took same data points as defined in table 1 in both cases and got the result as defined in Fig. 2. Advantage of this new technique over the ordinary techniques is that in the novel technique we are not taking higher order derivatives to find unknown constants which appeared in cubic spline polynomial whereas, in ordinary cubic spline higher order derivatives are used to find those unknown constants as shown in Fig. 2.

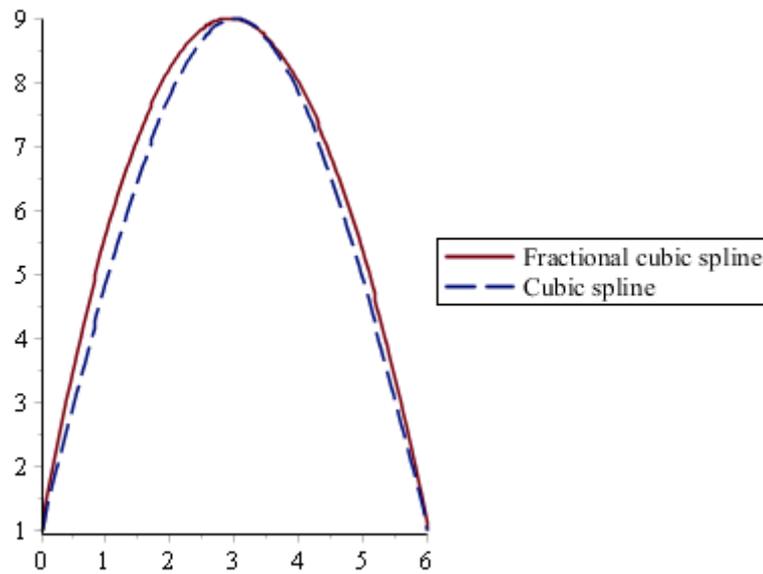


Fig.2 Comparison between ordinary cubic and fractional cubic spline.

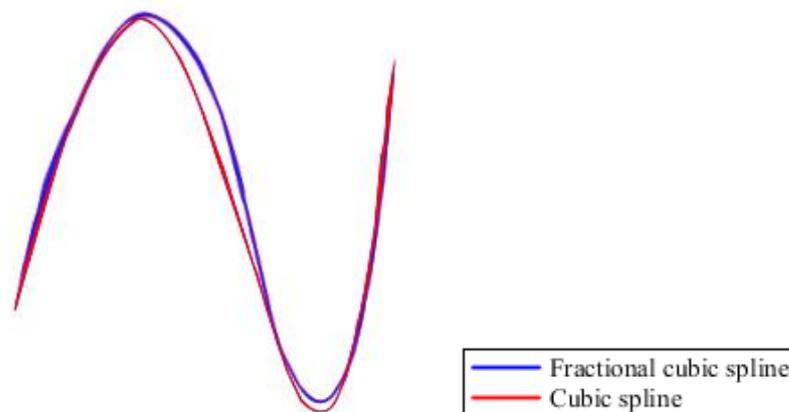


Fig.3 Comparison between cubic and Fractional spline using convex and concave data of Table 3

Table 3: Fractional order derivatives for α

X	0	3	6	8	9
Y	1	9	1	0	8

Fig. 3 depicts the consideration about four pieces of fractional cubic spline and ordinary cubic spline. Again the results show that both techniques are producing almost same results.

4 Effect of Order of Fractional Derivative and s and t on the Final Result

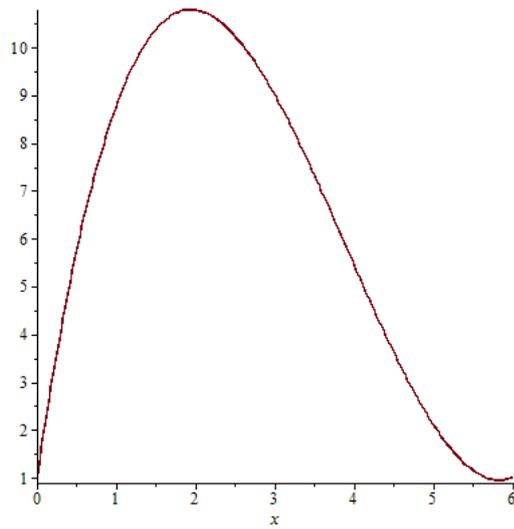


Fig.4 When s is close to x_i and t is close to x_{i+2}

The two parameters s and t which are introduced in formula can also use to control the shape of the curve which is quite evident in Fig. 4 when the parameter s is very close to x_i and t is close to x_{i+2} the curve is tilted.

The order of fractional derivative will also affect the shape of the curve. Although, it will pass through the same data point but the effect of final shape is quite evident from the following results.

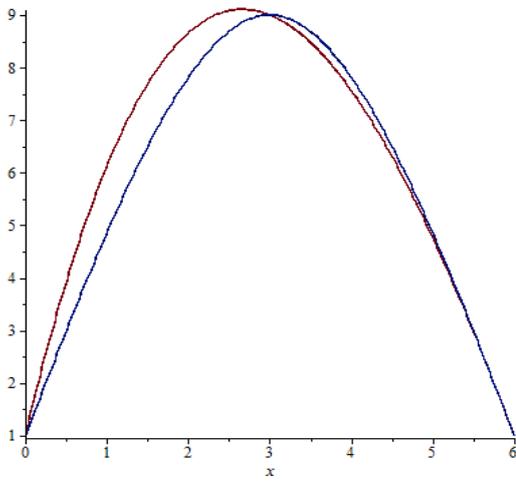


Fig. 5(a)

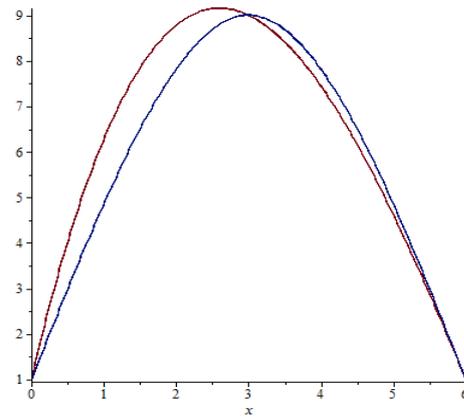


Fig. 5(b)

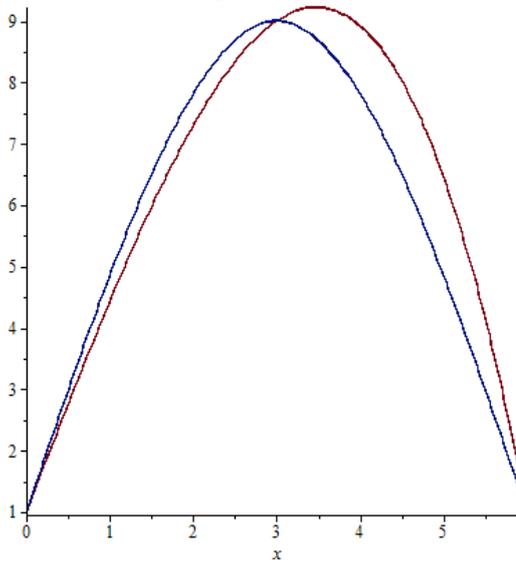


Fig. 5(c)

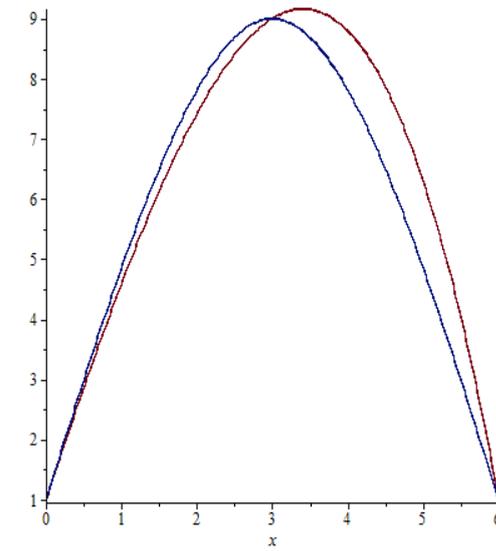


Fig. 5(d)

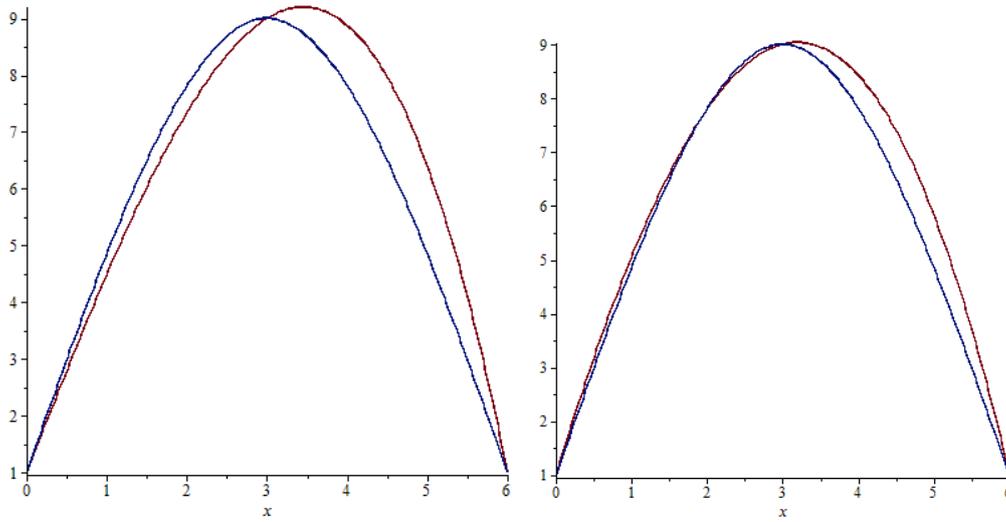


Fig. 5(e)

Fig. 5(f)

Table 4: Fractional order derivatives for different α values

α	Fig. 5(a)	.90	.92	.95
	Fig. 5(b)	.70	.92	.95
	Fig. 5(c)	.70	.80	.95
	Fig. 5(d)	.50	.60	.70
	Fig. 5(e)	.50	.55	.65
	Fig. 5(f)	.40	.50	.60

We assigned different values to fractional derivative α which are given in table 4. Different extensions of fig. 5 show the affect of these fractional derivatives on shape of the curve.

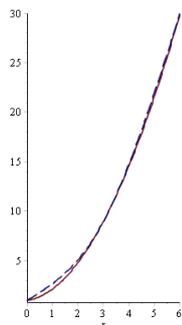


Fig. 6(a)

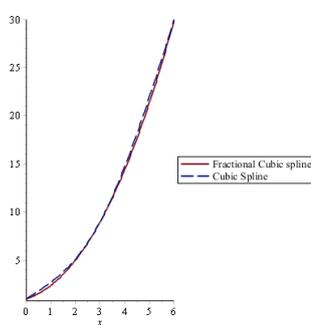


Fig. 6(b)

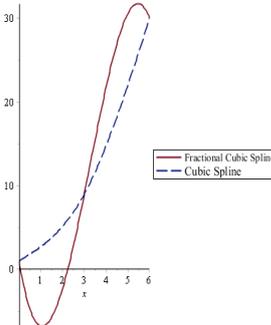


Fig. 6(c)

Table 5: Fractional order derivatives for different α values

α	Fig. 6(a)	.90	.92	.95
	Fig. 6(b)	.70	.80	.95
	Fig. 6(c)	.60	.70	.80

Table 5 illustrates how fractional order α control the path of piecewise curves. The aforementioned information shows that when the $0.7 < \alpha < 1$, both ordinary cubic spline and fractional order cubic spline almost follow the same path. When $\alpha < 0.7$, fractional cubic spline astray from the original path. Due to this very reason, we have a better control on the curve and we can mold the path as needed.

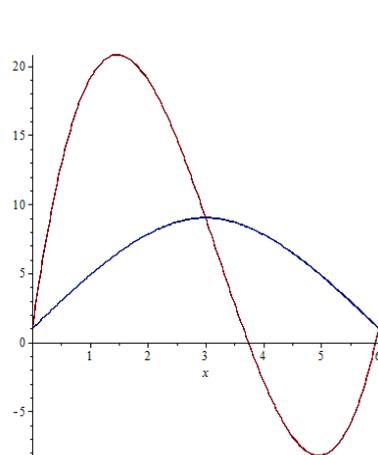


Fig. 7(a)

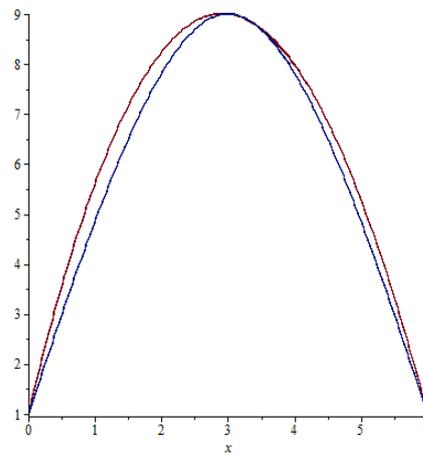


Fig. 7(b)

Fig. 7(a) shows if the fractional derivatives are less than 0.5 and if the shape parameters s and t are close to the tabulated values, the curve will not be bound to follow the right track. Whereas the Fig. 7(b) shows if s and t got the same values as that of Fig. 7(a) but the fractional derivatives got values as defined in Table 2, the fractional cubic spline will retain the right track. Therefore, not only α but s and t also plays a pertinent role in final shape that means we have more liberty than that of ordinary piecewise cubic spline.

5. Conclusion

Aforementioned results showed that newly introduced piecewise cubic fractional order spline technique is reliable and can become an effective alternate for the old techniques. It was also evident from the study that in the novel technique, we can locally control the piecewise curves either by fluctuating α or by changing the two introduced parameters of s and t while keeping the tabulated values intact. This

also shows that the piecewise curves will definitely pass through the given data points and still we can somewhat control the final shape of the curve.

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